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Investigation of Nonideal Plasma Properties

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CONTENTS

		Page
I.	INTRODUCTION	1
II.	NONTHERMAL IONIZATION IN ALKALI PLASMAS	3
III.	EXCHANGE ENERGY OF DENSE PLASMAS	13
IV.	STATISTICAL IONIZATION EQUILIBRIUM IN PARTIALLY DEGENERATE PLASMAS	25
V.	INTEGRAL TABLES FOR THERMODYNAMIC FUNCTIONS OF FERMION SYSTEMS	41
VI.	PROPAGATION OF PROBABILITY DENSITY IN QUANTUM MECHANICS	97
VII.	APPENDIX: INCOMPRESSIBLE MHD FLOW WITH HEAT TRANSFER BETWEEN INCLINED WALLS	121

INTRODUCTION

ONR Contract N00014-78-C-0117 is concerned with the calculation of the properties of nonideal plasmas. The research reported herein has been conducted by H. E. Wilhelm, S. H. Hong and S. H. Choi.

In Chapter II, a simple theory of the pressure ionization of plasmas at densities where the wave functions of the atoms begin to overlap is given. It is shown that complete (single) ionization of alkali plasmas can be obtained by means of pressure ionization at pressures achievable by means of schock waves driven by explosives. The anomalous high electrical conductivity of high-pressure plasmas is mainly due to pressure ionization, whereas the momentum transfer (collision time) is similar to ordinary plasmas. In Chapter III, an analytical expression for the exchange energy of partially degenerate plasmas is given, while the calculation of the electron-ion interaction energy for strongly nonideal plasmas could not be completed in time. The exchange energy represents an important contribution to the free energy of a nonideal plasma, since it is of the same order-of-magnitude as the thermal energy in low temperature, high pressure plasmas. In Chapter IV, the Saha equation for partially degenerate plasmas is discussed. It is demonstrated that the ionization equilibrium is practically not changed by degeneracy at pressures below pressure ionization. In Chapter V, the integrals of the thermodynamic functions of electron gases of arbitrary degeneracy are computed. This computer program was required for the quantitative determ ination of the degeneracy effects, and chemical potential as a function of density. In Chapter VI, the "diffuse" propagation of wave packets with infinite speed within the conventional, parabolic quantum theory is discussed. The need for a hyperbolic theory which describes discontinuous

wave packets with finite wave front speeds is stressed, since the acceleration of electrons (wave packets) in the electric field within dense plasmas can not be calculated properly by means of the Schrödinger equation. In the Appendix VII, a MHD flow analysis initiated during the previous ONR Contract is reported.

This report contains first, preliminary results, which will be submitted in final form as publications at a later date.

II. NONTHERMAL IONIZATION IN ALKALI PLASMAS

ABSTRACT

Approximate formulae are derived for the critical density and pressure at which the atoms of hydrogen-like plasmas become ionized due to overlapping of the wave functions. It is shown that by this mechanism not only the thermally excited but also the ground state atoms of alkali plasmas become ionized already at moderate pressures. Numerical examples are given for H, Li, Na, K, Rb, and Cs plasmas.

The anomalous high ionization observed in nonideal plasmas at high pressures and low to moderate temperatures appears to be due to nonthermal pressure ionization, which becomes significant as the average distance between atoms approaches the order of the atomic diameter, $2a_o \sim 10^{-8}$ cm. This quantum mechanical ionization occurs as result of the overlapping of the atomic wave functions at sufficiently high pressures and any temperature $T \geq 0$ not only in gaseous but also in solid media.[1)]

In an isothermal plasma, the degree of ionization κ decreases first with increasing pressure P (region of thermal ionization), reaches a minimum, and then steeply increases with a further raise of P as the wave functions of the atoms begins to overlap more or less suddenly (Fig. 1). The electrical conductivity $\sigma = (ne^2/m)\tau$ of an isothermal plasma shows a similar P-dedpendence, since the electron density n varies similar to κ with P and the momentum relaxation time τ varies relatively little with P(Fig. 1).

The $\sigma(P)$ minimum appears to be promising for the operation of high pressure plasmas. An electrical discharge initiated at a pressure P in the vicinity of the $\sigma(P)$ minimum, should spread throughout the entire system volume since $d\sigma = (\partial\sigma/\partial P)_{min}\, dP \approx 0$ everywhere including the region of the respective current transport where $dP > 0$ due to heating and ionization (whereas $d\sigma > 0$ in an ordinary arc channel and $d\sigma \approx 0$ outside of it).

The exact calculation of the ionization in a high pressure plasma considering thermal excitation and ionization, quantum-mechanical ionization due to the overlapping of the wave functions of the atoms in ground and excited states, and the changes of the atomic levels due to the strong

nonideal interactions in the plasma represents a formidable theoretical and computational effort[2]. For this reason, we develop in this note simple formulae for the estimation of the critical density and pressure at which nonthermal pressure ionization occurs.

In view of the practical importance of alkali plasmas, the calculations are carried through for hydrogen-like systems. In the numerical illustrations, the onset of quantum mechanical pressure ionization in H, Li, Na, K, Rb, and Cs plasmas is discussed within the frame of the approximations made.

PRESSURE IONIZATION

The overlapping of the atomic wave functions in plasmas occurs at such high pressures that the electron gas exhibits degeneracy effects. The average kinetic energy $\bar{E}$ of a free electron resulting from pressure ionization is according to Fermi statistics[3)]

$$\bar{E} = \frac{3}{2} KT\, U_{3/2}(\mu/KT)/U_{1/2}(\mu/KT) \tag{1}$$

where

$$U_\rho(\mu/KT) = \frac{1}{\Gamma(\rho+1)} \int_o^\infty \frac{x^\rho\, dx}{e^{x-\mu/KT}+1} \tag{2}$$

and the chemical potential μ is given in terms of the electron density n and temperature T by the integral functional,

$$n = 2(2\pi m\, KT/h^2)^{3/2}\, U_{1/2}(\mu/KT) \tag{3}$$

The binding energy of a hydrogen-like atom with charge number Z in a state with principal quantum number n and orbital quantum number ℓ is[4)]

$$E_{n\ell} = - \hbar^2/2m\, a_{n\ell}^2 \tag{4}$$

where

$$a_{n\ell} = \frac{\hbar^2}{Zme^2}[n - \alpha(\ell)] \tag{5}$$

is the atomic "radius" in state (n, ℓ). Accordingly, the energy required for the ionization of a hydrogen-like atom in state (n, ℓ) is

$$\Delta E_{n\ell} = (\hbar^2/2m)a_{n\ell}^{-2} \quad . \tag{6}$$

From the condition $\bar{E} = \Delta E_{n\ell}$ we deduce the following relation for the critical electron density n at which the atomic component in the excited state n is completely ionized due to overlapping of the atomic wave functions at a given temperature T:

$$\frac{3}{2} KT \frac{U_{3/2}(\mu/KT)}{U_{1/2}(\mu/KT)} = \frac{\hbar^2}{2ma_{n\ell}^2} \qquad (7)$$

where $\mu = \mu(n, T)$ by Eq.(3). The critical electron pressure p for pressure ionization is given in terms of the critical electron density n by

$$p = nKT[U_{3/2}(\mu/KT)/U_{1/2}(\mu/KT)]. \qquad (8)$$

For complete degeneracy of the free electron gas, Eqs. (1), (7) and (8) reduce to:

$$\bar{E} = (3/10)\pi^2(3/\pi)^{2/3}(\hbar^2/m)n^{2/3} \quad , \quad n >> \tilde{n} \quad , \qquad (9)$$

$$n = (1/3\pi^2)(5/3)^{3/2} a_{n\ell}^{-3} \quad , \quad n >> \tilde{n} \quad , \qquad (10)$$

$$p = (1/15\pi^2)(5/3)^{5/2}(\hbar^2/m)a_{n\ell}^{-5} \quad , \quad n >> \tilde{n} \quad , \qquad (11)$$

where

$$\tilde{n} = 2(2\pi mKT/h^2)^{3/2} = 4.828 \times 10^{15} T^{3/2} [cm^{-3}] . \qquad (12)$$

is the characteristic electron density at which degeneracy becomes significant. Numerically, Eqs. (10) and (11) yield

$$n = 7.267 \times 10^{-2} a_{n\ell}^{-3} [cm^{-3}] \quad , \qquad (13)$$

$$p = 2.957 \times 10^{-29} a_{n\ell}^{-5} [dyne/cm^2] . \qquad (14)$$

It should be noted that $n \propto a_{n\ell}^{-3}$ and $p \propto a_{n\ell}^{-5}$ depend strongly on the atomic radius $a_{n\ell}$. Since in general the atomic radius $a_{n\ell}$ of an excited atom is larger than that of an atom in the ground state, the excited atoms will be ionized at a lower pressure. Thus, pressure ionization is an important mechanism for the explanation of the anomalous high ionization in plasmas of moderately high pressures.

APPLICATION

Let the above results be applied to low temperature plasmas, $KT \ll \varepsilon_1$, where ε_1 is the excitation energy of the first state above the ground state of the atoms. In TABLE 1, numerical values of the critical density n and pressure p are given for hydrogen and alkali plasmas at a temperature $T = 10^3\ {}^oK$ based on Eqs. (7) and (8) and empirical data for the radii a_o of the atoms with interaction[5)]. In TABLE 2, the corresponding numerical values are given based on Eqs. (10) and (11), which assume complete degeneracy of the free electrons. It is seen that, in the cases considered, thermal effects are not very important so that the critical values of n and p can be estimated from the simpler, explicit formulae in Eqs. (10) and (11).

Tables 1 and 2 indicate that the small hydrogen atoms are least promising and the large cesium atoms are most attractive for nonideal plasma experiments. The critical electron densities n of the plasma state are by not quite one order of magnitude smaller than the free electron densities of the corresponding (solid) metals, e.g. $n = 0.78 \times 10^{24}\ cm^{-3}$ in metallic hydrogen and $n = 2.58 \times 10^{22}\ cm^{-3}$ in metallic cesium. The critical electron pressures decrease from $p \sim 7 \times 10^6$ bar for H to $p \sim 2 \times 10^3$ bar for Cs.

It should be noted that the electron pressure p does in general not give the correct order of magnitude of the total plasma pressure P, since $P < p$ due to the (negative) electron-ion interaction energy and electron exchange energy. The latter nonideal effects are most pronounced at high electron densities, e.g., in the example of the hydrogen plasma ($n \sim 5 \times 10^{23}\ cm^{-3}$).

TABLE 1: Critical values of n and p for $T = 10^3$ °K based on Eqs. (7) and (8).

Element	a_o[cm]	n[cm^{-3}]	p[bar]
H	0.67×10^{-8}cm	2.36×10^{23}	2.18×10^{6}
Li	1.58×10^{-8}cm	1.83×10^{22}	2.98×10^{4}
Na	1.82×10^{-8}cm	1.19×10^{22}	1.46×10^{4}
K	2.22×10^{-8}cm	6.46×10^{21}	5.33×10^{3}
Rb	2.43×10^{-8}cm	4.86×10^{21}	3.35×10^{3}
Cs	2.60×10^{-8}cm	3.92×10^{21}	2.36×10^{3}

TABLE 2: Critical values of n and p for $T \to 0$ based on Eqs. (10) and (11).

Element	a_o[cm]	n[cm^{-3}]	p[bar]
H	0.67×10^{-8}cm	2.37×10^{23}	2.19×10^{6}
Li	1.58×10^{-8}cm	1.84×10^{22}	3.00×10^{4}
Na	1.82×10^{-8}cm	1.20×10^{22}	1.48×10^{4}
K	2.22×10^{-8}cm	6.64×10^{21}	5.48×10^{4}
Rb	2.43×10^{-8}cm	5.06×10^{21}	3.49×10^{3}
Cs	2.60×10^{-8}cm	4.13×10^{21}	2.49×10^{3}

Since the atomic radius $a_{n\ell}$ increases with increasing level number n, the thermally excited atoms are much larger than those in the ground state. For this reason, pressure ionization will occur at considerably lower pressures in plasmas of moderate temperatures, $KT \gtrsim \varepsilon_1$. This qualitative conclusion appears to be confirmed by experiments in Cs plasmas at a temperature of T=2500 ^{o}K which show a steep increase of the electrical conductivity at a pressure of only $P \simeq 10^2$ atm[6)].

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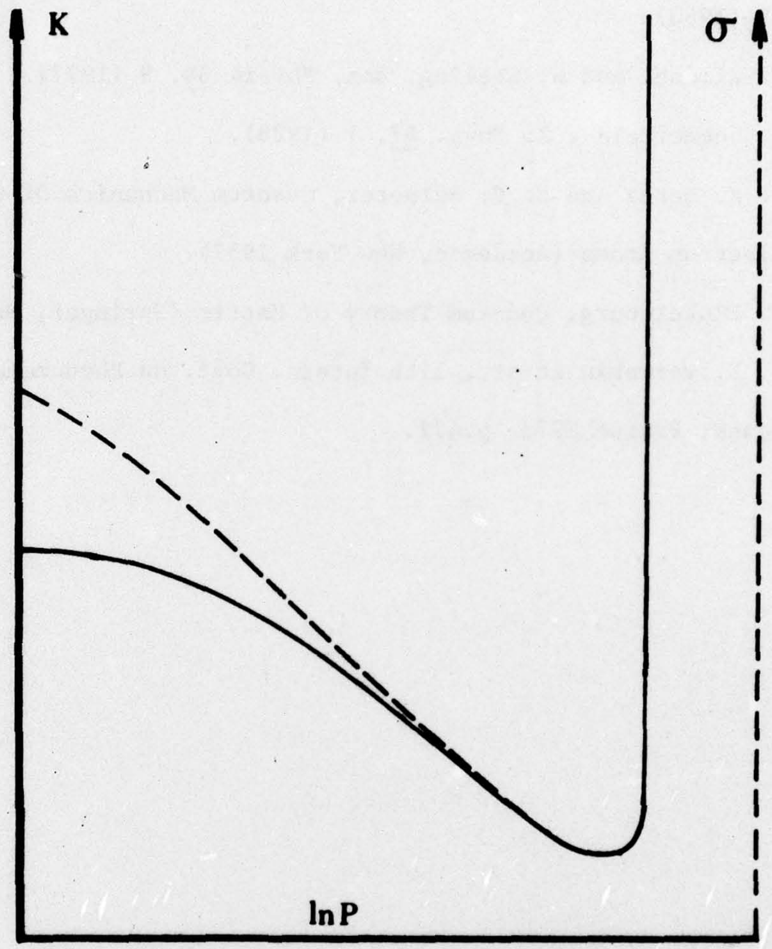

FIG. 1: Degree of ionization κ and electrical conductivity σ versus P (qualitative with different scales for κ and σ minima to coincide).

III. EXCHANGE ENERGY OF DENSE PLASMAS

ABSTRACT

The quantum-mechanical exchange energy is calculated for plasmas with Fermi and Maxwell momentum distributions of the electrons. It is shown that at not too high temperatures T, the (negative) exchange energy is of the order of magnitude of the thermal energy of the particles if the electron density is sufficiently high. The required electron densities are achievable by nonthermal pressure ionization (overlapping of the atomic wave functions).

Plasmas of relatively low temperatures, 10^3 $^oK < T < 10^5$ ^oK, but extremely high pressures, 10^2 Bar $\lesssim T \lesssim 10^6$ Bar, are technically important as thermal energy sources for various unclassified and classified applications. Above a critical pressure, e.g. $P_o \simeq 2.5 \times 10^3$ Bar for cesium, all atoms in plasmas are at least singly ionized due to the overlapping of the atomic wave functions.[1] The large electron densities of plasmas at pressures $P > P_o$ cause anomalously high electrical conductivities which approach values comparable to those of metals ($\sigma \sim 5 \times 10^5$ mho/cm).

It is known that metals below their melting point are held together in form of a solid by the negative electron-ion interaction energy and the negative exchange of the free electrons.[2] Since the Fermi energy, $E_F = (\hbar^2/2m)(3n/8\pi)^{2/3}$, corresponds to temperatures of the order $T_F \sim 5 \times 10^4$ ^oK for metals ($n \sim 5 \times 10^{22}$ electrons/cm^3), the exchange energy E_A of metals can be calculated without taking thermal effects into account.[1] The exchange energy per electron in metals is of the order $-E_A \sim 5$ eV.

In a dense plasma, the (negative) electron-ion interaction energy and the exchange energy of the electron gas have the effect of a "binding energy" whereas the (positive) thermal energy tends to spread the plasma out in space by random motions. For a given temperature T, the electron-ion interaction energy density $\varepsilon_c(n,T)$ and the exchange energy density $\varepsilon_A(n,T)$ of the plasma equals in magnitude its thermal energy density $\varepsilon_T(n,T)$ at electron densities n above a critical value $n_c = n_c(T)$. For $n > n_c(T)$, the plasma is in a cohesive state similar to that of a metal.

The exchange energy density $\varepsilon_A(n,T)$ of dense plasmas is evaluated under consideration of thermal effects for Fermi (low temperatures) and Maxwell

(high temperatures) energy distributions of the electrons. An attempt of solving this problem has been made recently by Glasser[3], who obtains the zero order and first order terms of the exchange energy density in a series expansion for low temperatures by representing the Fermi distributions in the exchange integral as two-sided inverse Laplace transforms. According to this result,[3] the logarithmic temperature dependence of the exchange energy density in the low temperature limit found by Kojima and Isihara[4] appears to be incorrect.

We could not yet derive an analytical expression for the e-i interaction energy which is of sufficient accuracy for strong nonideal conditions. This problem will be the subject of future work.

EXCHANGE ENERGY

The quantum mechanical exchange effect exists for Fermi systems, i.e. for systems of N particles the wave function $\Psi = \psi(\vec{x}_1, \vec{x}_2, \ldots \vec{x}_N)$ of which is antisymmetric against the exchange of the space-spin coordinates $\vec{x}_i$ and $\vec{x}_j$ of any particle pair ij (Pauli principle). For this reason, the exchange energy of a plasma is due to its electrons of spin $s = \pm \hbar/2$ and mass m. The contribution to the exchange energy from ionic (i) and atomic (a) Fermi components is negligible since $m_{i,a} >> m$.

The total Coulomb interaction energy of a plasma consists of the (negative) electron-ion interaction energy, the ion-ion and the electron-electron interaction energies. The negative Coulomb interaction energy of the electron gas due to the quantum mechanical exchange effect is given by the double sum over electron pairs ij of parallel spins[2]:

$$E_A = \frac{1}{2} \sum_{\substack{i=1 \\ i \neq j}}^{N} \sum_{j=1}^{N} E_{ij} \tag{1}$$

where

$$E_{ij} = - \int_\Omega \int_\Omega \frac{e^2/4\pi\varepsilon_0}{|\vec{r}_2 - \vec{r}_1|} \psi_i^*(\vec{r}_1)\psi_i(\vec{r}_2)\psi_j^*(\vec{r}_2)\psi_j(\vec{r}_1) d^3\vec{r}_1 d^3\vec{r}_2 \tag{2}$$

is the exchange energy per electron pair ij of parallel spins (electron pairs with antiparallel spin have zero exchange energy). Eqs. (1) - (2) give the exchange energy of the plasma in the 0-th approximation[1] which assumes that the wave function $\Psi(\vec{r}_1, \vec{r}_2), \ldots \vec{r}_N)$ of the electron gas can be represented by a Slater determinant of one-electron wave functions $\psi_i(\vec{r})$, i = 1, 2, ... N, of the form of plane waves of momenta $\vec{p}_i = \hbar\vec{k}_i$,

$$\psi_i(\vec{r}) = \Omega^{-\frac{1}{2}} e^{i\vec{k}_i\cdot\vec{r}} \quad . \tag{3}$$

The resulting density $n(\vec{r})$ of electrons in the plasma of volume Ω is homogeneous,

$$n(\vec{r}) = \sum_{i=1}^{N} \psi_i^*(\vec{r})\psi_i(\vec{r}) = N/\Omega \quad . \tag{4}$$

Substitution of Eq.(3) into Eq.(2) and spatial integrations given for the exchange energy of an electron pair ij,

$$E_{ij} = -\frac{e^2/\Omega\,\varepsilon_o}{(\vec{k}_i - \vec{k}_j)^2} \quad . \tag{5}$$

The number $dN^{\uparrow\downarrow}$ of electrons with spin $s = +\hbar/2$ ($\uparrow$) or spin $s = -\hbar/2$ ($\downarrow$) in the $\vec{k}$-space volume element $d^3\vec{k}$ is

$$dN^{\uparrow\downarrow} \equiv d(N/2) = \Omega f(\vec{k})\; d^3\vec{k} \tag{6}$$

where

$$f(\vec{k}) = \frac{(2\pi)^{-3}}{e^{\alpha\vec{k}^2-\beta\mu}+1} \quad , \tag{7}$$

$$\alpha = \hbar^2/2mKT, \qquad \beta = 1/KT \quad , \tag{8}$$

for a Fermi distribution of electron momenta $\vec{p} = \hbar\vec{k}$, and

$$f(\vec{k}) = ae^{-b\vec{k}^2} \quad , \tag{9}$$

$$a = (m/2\pi KT)^{3/2}(\hbar/m)^3 N/2\Omega \quad , \quad b = (m/2KT)(\hbar/m)^2 \quad , \tag{10}$$

for a Maxwell distribution of electron momenta $\vec{p} = \hbar\vec{k}$ (K = Boltzmann constant, h = Planck constant). In Eq.(7), the chemical potential μ of the electrons of either spin is given by the integral functional,[2)]

$$N^{\uparrow\downarrow} \equiv N/2 = \Omega(2\pi mKT/h^2)^{3/2}\,\Gamma^{-1}(3/2)\int_0^\infty \frac{x^{\frac{1}{2}}dx}{e^{x-\beta\mu}+1} \quad . \tag{11}$$

By menas of the distribution function $f(\vec{k})$ in Eq.(6), the i and j summations in Eq.(1) are replaced by integrals in $\vec{k}$-space. Thus, one finds under consideration of Eq.(5) the following expression for the exchange energy E_A of the N electrons of the plasma ($E_A = 2E_A^{\uparrow\downarrow}$):

$$E_A = -(e^2\Omega/\varepsilon_o) \int\!\!\int_{-\infty-\infty}^{+\infty+\infty} (\vec{k}_i-\vec{k}_j)^{-2} f(\vec{k}_i) f(\vec{k}_j) d^3\vec{k}_i \; d^3\vec{k}_j \quad . \tag{12}$$

The exchange energy is readily evaluated in closed form for metals with $KT<<E_F$. The evaluation of the exchange energy of a plasma is mathematically more involved since the thermal effects have to be taken into consideration. It will be seen that the exchange energy of a plasma varies significantly with temperature T and the type of momentum distribution. The electrons exhibit i) a Fermi or ii) a Maxwell distribution depending on whether their density is i) $n \gtrsim \tilde{n}$ or ii) $n<<\tilde{n}$, where[2)]

$$\tilde{n} = 2(2\pi mKT/h^2)^{3/2} \quad . \tag{13}$$

1. E_A for Fermi Plasma

Noting that $d^3\vec{k}_i = 2\pi k_i{}^2 dk_i \sin\theta \, d\theta$, $\theta = \sphericalangle(\vec{k}_i,\vec{k}_j)$, and $d^3\vec{k}_j = 4\pi k_j{}^2 dk_j$, substitution of Eq. (7) for $f(\vec{k})$ into Eq.(12) yields for the exchange energy of a Fermi electron plasma

$$E_A = -\frac{e^2 \; \Omega}{4\pi\varepsilon_o 2\pi^3} \int\!\!\int\!\!\int_{ooo}^{\pi\infty\infty} \frac{\sin\theta \; d\theta \; k_i{}^2 dk_i \; k_j^2 dk_j}{\left(k_i^2+k_j^2-2k_ik_j\cos\theta\right)\left(e^{\alpha k_i^2-\beta\mu}+1\right)\left(e^{\alpha k_j^2-\beta\mu}+1\right)} \tag{14}$$

where

$$\int_o^{\pi} \left(k_i^2+k_j^2-2k_ik_j\cos\theta\right)^{-1} \sin\theta \; d\theta = k_i^{-1}k_j^{-1} \; \ell n \left[\left(k_i+k_j\right)/|k_i-k_j|\right]. \tag{15}$$

is singular at $k_i = k_j$. Introduction of polar coordinates,

$$k_i = \rho\cos\phi \;, \quad k_j = \rho\sin\phi \;, \quad dk_i \; dk_j = \rho d\rho \; d\phi \quad , \tag{16}$$

where $0 \leq \phi \leq \pi/2$ since $k_{i,j} \geq 0$, transforms Eq.(14) to

$$E_A = -\frac{e^2\Omega}{4\pi\varepsilon_o 8\pi^2} \int_o^{\pi/2} \int_o^{\infty} \frac{\mathcal{D}(\phi) \; d\phi \;\; \rho^3 \; d\rho}{\left(e^{\alpha\rho^2\cos^2\phi-\beta\mu}+1\right)\left(e^{\alpha\rho^2\sin^2\phi-\beta\mu}+1\right)} \tag{17}$$

where

$$\mathcal{D}(\phi) = (4/\pi) \sin\phi \cos\phi \; \ell n[(\cos\phi + \sin\phi)/|\cos\phi - \sin\phi| \;] \quad , \tag{18}$$

$$\mathcal{D}(\phi=\pi/4) = \infty \ , \qquad \int_0^{\pi/2} \mathcal{D}(\phi)\, d\phi = 1 \qquad . \tag{19}$$

Since $\mathcal{D}(\phi)$ is a quasi-Dirac function with a singularity at $\phi = \pi/4$, Eq.(17) becomes after ϕ-integration

$$E_A \cong - \frac{(e^2/4\pi\varepsilon_o)}{(4\pi)^2}\,\Omega \int_0^\infty \frac{\rho^2\, d\rho^2}{\left(e^{\frac{1}{2}\alpha\rho^2-\beta\mu}+1\right)^2} \qquad . \tag{20}$$

Series expansion of the integral in Eq.(20) for partially degenerate ($e^{\beta\mu}<1$) conditions yields for the exchange energy of the plasma:

$$E_A \cong -\left(\frac{e^2}{4\pi\varepsilon_o}\right)\frac{e^{2\beta\mu}\,\Omega}{\left(4\pi\alpha\right)^2}\sum_{\nu=0}^{\infty}(-1)^\nu(\nu+1)\left(\frac{2}{\nu+2}\right)^2\left(e^{\beta\mu}\right)^\nu, \quad e^{\beta\mu}<1, \tag{21}$$

or

$$E_A \cong -\left(\frac{e^2}{4\pi\varepsilon_o}\right)\frac{e^{2\beta\mu}\,\Omega}{\left(4\pi\alpha\right)^2}\left[1-\frac{8}{9}\left(e^{\beta\mu}\right)^1+\frac{12}{16}\left(e^{\beta\mu}\right)^2-\frac{16}{25}\left(e^{\beta\mu}\right)^3+\ldots\right], \quad e^{\beta\mu}<1. \tag{22}$$

The chemical potential $\mu = \mu(n,T)$ is determined by Eq.(11), which reduces for partially degenerate ($e^{\beta\mu}<1$) conditions to[2)]

$$n = 2(2\pi mKT/h^2)^{3/2}\, e^{\beta\mu}\sum_{\nu=0}^{\infty}(-1)^\nu(1+\nu)^{-3/2}\,(e^{\beta\mu})^\nu, \quad e^{\beta\mu}<1, \tag{23}$$

or

$$n = 2(2\pi mKT/h^2)^{3/2}\, e^{\beta\mu}[1-2^{-3/2}(e^{\beta\mu})^1+3^{-3/2}(e^{\beta\mu})^2-4^{-3/2}\,(e^{\beta\mu})^3+\ldots] \ , \quad e^{\beta\mu}<1. \tag{24}$$

Inversion of the series (24) gives the chemical potential $\mu = \mu(n,T)$ explicitly in the same approximation,

$$\mu = \beta^{-1}\,\ell n\left\{\frac{n}{\tilde{n}}\left[1+2^{-3/2}\left(\frac{n}{\tilde{n}}\right)^1+\left(\frac{1}{4}-3^{-3/2}\right)\left(\frac{n}{\tilde{n}}\right)^2 + \left(\frac{1}{8}+5\cdot 2^{-9/2}-5\cdot 6^{-3/2}\right)\left(\frac{n}{\tilde{n}}\right)^3+\ldots\right]\right\} \ , \quad n<\tilde{n} \quad . \tag{25}$$

where the critical density $\tilde{n}$ is defined in Eq. (13).

In general, gaseous as hot plasmas are in a partially degenerate state at high pressures $P \lesssim 10^6$ Bar, i.e. $e^{\beta\mu} < 1$ or $n < \tilde{n}$. For these conditions, the exchange energy and chemical potential in Eqs. (21)-(25) are applicable. In the classical limit, $e^{\beta\mu} << 1$, Eqs.(22) and (25) result in simple formulae for the exchange energy and chemical potential,

$$E_A = -\frac{e^2}{4\pi\varepsilon_o}\frac{\Omega}{(4\pi\alpha)^2}\left(\frac{n}{\tilde{n}}\right)^2, \qquad n<<\tilde{n} \quad , \tag{26}$$

$$\mu = \beta^{-1}\,\ell n(n/\tilde{n}), \qquad n<<\tilde{n} \, . \tag{27}$$

2. E_A for Maxwell Plasma

With $d^3\vec{k}_i = 2\pi k_i^2\, dk_i \sin\theta\, d\theta$, $\theta = \sphericalangle(\vec{k}_i,\vec{k}_j)$, and $d^3\vec{k}_j = 4\pi k_j^2\, dk_j$, substition of Eq.(9) for $f(\vec{k})$ into Eq.(12) yields for the exchange energy of a Maxwell electron plasma

$$E_A = -32\pi^3\left(\frac{e^2}{4\pi\varepsilon_o}\right)\Omega a^2 \int_0^\pi\int_0^\infty\int_0^\infty \frac{e^{-b(k_i^2+k_j^2)}\sin\theta\, d\theta\, k_i^2\, dk_i\, k_j^2\, dk_j}{(k_i^2+k_j^2-2k_ik_j\cos\theta)} \quad . \tag{28}$$

Intergration with respect to θ and introduction of polar coordinates (16) reduces Eq.(28) to

$$E_A = -16\pi^3\left(\frac{e^2}{4\pi\varepsilon_o}\right)\Omega a^2 \int_0^{\pi/2} \sin\phi\cos\phi\,\ell n\left(\frac{\cos\phi+\sin\phi}{|\cos\phi-\sin\phi|}\right)d\phi \int_0^\infty e^{-b\rho^2}\rho^2 d\rho^2 \, . \tag{29}$$

Evaluation of the ϕ-integral [Eqs. (18)-(19)] and the Euler integral yields for the exchange energy of the classical plasma the formula

$$E_A = -4\pi^4\,(e^2/4\pi\varepsilon_o)\Omega(a/b)^2 \quad , \tag{30}$$

or

$$E_A = -\frac{\pi}{2}\left(\frac{e^2}{4\pi\varepsilon_o}\right)\frac{\hbar^2}{m}\,\Omega\,\frac{n^2}{KT} \quad . \tag{31}$$

upon replacing a and b in accordance with Eq.(10). It is recognized that the exchange energy density $\varepsilon_A = E_A/\Omega$ is proportional to n^2T^{-1} in the classical limit.

Eq.(30) or Eq.(31) are mathematically exact results for the classical or nondegenerate plasma within the plane wave approximation. It is remarkable that these equations agree with Eq.(26), which was derived by means of an approximate evaluation of the ϕ-integral in Eq.(17), corresponding to an expansion around the singularity of $\mathcal{D}(\phi)$, Eq.(18). Evaluations of the exchange energy density based on the place waves, Eq.(5) but a modified Fermi distribution containing in addition to the plane wave energy $\varepsilon = (\hbar\vec{k})^2/2m$ also the exchange energy of the electron[5] are theoretically inconsistent, i.e. represent a questionable improvement over the consistent use of the plane wave approximation. A more rigorous calculation of the exchange energy would require an exact evaluation of the wave function of the many-particle plasma systems under consideration of all particle interactions, and subsequent integration of the exchange integral for this wave function.

APPLICATION

The quantitative importance of the exchange energy density for high pressure plasmas becomes obvious if one compares it with the thermal energy density ε_T of the electrons. According to Fermi statistics[2],

$$\varepsilon_T = \frac{3}{2} nKT\ h(\beta\mu) \tag{32}$$

where

$$h(\beta\mu) = U_{3/2}(\beta\mu)/U_{1/2}(\beta\mu), \quad U_\rho(\beta\mu) = \Gamma^{-1}(1+\rho)\int_o^\infty \frac{x^\rho dx}{(e^{x-\beta\mu}+1)} \tag{33}$$

or

$$h(\beta\mu) = 1 + \frac{\sqrt{2}}{8}(e^{\beta\mu})^1 - \frac{1}{16}\left(\frac{32}{27}\sqrt{3} - 1\right)(e^{\beta\mu})^2 + \frac{1}{32}\left(3 + \frac{\sqrt{2}}{2} - \frac{28}{27}\sqrt{6}\right)(e^{\beta\mu})^3 + \ldots, \quad e^{\beta\mu} < 1. \tag{34}$$

considers the degeneracy effects. The exchange energy density,

$$\varepsilon_A = E_A/\Omega \tag{35}$$

is given by Eqs. (20) and (21) for arbitrary and partial degeneracy, respectively. The series expansions of Eqs. (21) and (34) indicate that the nondegenerate ($\exp\beta\mu<<1$) expressions for ε_A and ε_T give the correct order of magnitude even in the case of partial degeneracy ($\exp\beta\mu<1$). For this reason, it is sufficient to compare the energy densities in the limit $n << \tilde{n}$:

$$\varepsilon_T = \frac{3}{2} nKT = 2.07 \times 10^{-16}\ nT\ [\mathrm{erg\ cm}^{-3}] \quad , \tag{36}$$

and

$$\varepsilon_A = -\frac{\pi}{2}\left(\frac{e^2}{4\pi\varepsilon_o}\right)\frac{\hbar^2}{m}\frac{n^2}{KT} = -\ 3.204 \times 10^{-30}\ \frac{n^2}{T}\ [\mathrm{erg\ cm}^{-3}]\ . \tag{37}$$

According to Eqs.(36) and (37), ε_T and ε_A are comparable in magnitude for conditions typical of low temperature, high pressure plasmas with pressure ionization due to the overlapping of the atomic wave functions.[1] E.g., $\varepsilon_T \sim 10^7$ erg cm^{-3} and $-\ \varepsilon_A \sim 10^7$ erg cm^{-3} for $T = 10^3$ °K and $n = 5 \times 10^{19}$ cm^{-3}; or $\varepsilon_T \sim 10^{10}$ erg cm^{-3} and $-\ \varepsilon_A \sim 10^{10}$ erg cm^{-3} for $T = 10^4$ °K and $n = 5 \times 10^{21}$ cm^{-3}.

For the n and T values in the examples, the plasma is only partially degenerate, $n \lesssim \tilde{n}$, where $\tilde{n} = 4.828 \times 10^{15} T^{3/2}\ cm^{-3}$, so that the formulae (36) and (37) can be used for order-of-magnitude estimates. For more accurate calculations the series expansions in Eqs. (34) and (21) have to be used.

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IV. STATISTICAL IONIZATION EQUILIBRIUM IN PARTIALLY DEGENERATE PLASMAS

Abstract

The thermal ionization equilibrium in plasmas is considered at pressures for which the electron gas is partially to considerably degenerate. An ionization equation is derived which takes into account that i) the electron energies are distributed according to Fermi statistics and ii) the (heavy) ions and atoms obey Boltzmann statistics, and which is valid up to pressures at which the wave functions of the atoms begin to overlap. A comparison of the quantum statistical and Saha ionization equations indicates that the degeneracy effects in the electron gas suppress somewhat the ionization. It is a remarkable result that the Saha equation describes, approximately, the thermal ionization equilibrium at extreme pressures (e.g., up to $P \approx 10^3$ Bar and $P \approx 10^6$ Bar in the cases of Cs and H plasmas, respectively), for which the electrons are noticeably degenerate.

In recent experiments, plasmas of relatively low temperatures ($10^3 < T < 10^5$ °K) but extremely high pressures ($10^2 < P < 10^6$ Bar) have been generated by shock wave compression.[1-4] Measurements indicate that the electrical conductivity $\sigma = (e^2/m)n_e\tau$ (e = elementary charge, m = electron mass) of high pressure plasmas is anomalously large, which has been attributed to nonideal plasma effects.[5-6] In partially ionized high pressure plasmas, for which the electron-neutral interactions are not negligible compared to the Coulomb interactions, the product $n_e\tau$ of electron density n_e and electron momentum relaxation time τ depends on both the electron (ion) density n_e and the atom density n_a. In this case, the evaluation of the electrical conductivity σ requires a quantitative determination of the electron (ion) and atom densities in the plasma for given pressure P and temperature, T.

For the pressure and temperature ranges under consideration, the electron component is partially degenerate. More quantitatively, a gas of Fermions of statistical weight g_s, mass m_s, and temperature T is partially degenerate at densities n_s larger than the critical value,[7]

$$\tilde{n}_s = g_s(2\pi m_s kT/h^2)^{3/2}$$

where k and h are the constants of Boltzmann and Planck. Fig. 1 shows $\tilde{n}_s$ versus T for electrons ($g_e = 2$, $m_e = 9.1096 \times 10^{-28}$ gr) and protons ($g_p = 2$, $m_p = 1.6726 \times 10^{-24}$ gr). For $T = 10^4$ °K, the electrons exhibit degeneracy effects already at densities $n_e > 10^{20}$ cm^{-3}, whereas degeneracy effects are negligible for protons and heavier Fermions in the pressure and temperature regions considered.

In the following, we give a quantum statistical derivation of the equation which describes the ionization-recombination equilibrium of

high-pressure plasmas, in which the electrons are partially to completely degenerate and the (heavy) ions and atoms behave classically. The relation of the quantum-statistical ionization equation to the classical Saha equation[8)] is discussed. As applications of the theory, the compositions of cesium and hydrogen plasmas are calculated for high pressures.

The quantitative results indicate that the quantum statistical ionization equation leads to a small depression of the ionization compared to the classical Saha ionization equilibrium for all conditions of thermal ionization, i.e., at all pressures up to the critical pressure at which the wave functions of neighboring atoms begin to overlap. Therefore, the Saha equation describes in some approximation the thermal ionization in plasmas at high pressures, even if the electrons are partially to considerably degenerate under these conditions.

The quantum statistical ionization-recombination equilibrium in external force fields shall not be treated here. Classical reactive systems in strong magnetic fields under equilibrium and nonequilibrium conditions have been treated by Wilhelm[9)] and Olivei[10)], respectively.

IONIZATION EQUATION

In the statistical equilibrium of a plasma, the microscopic ionization and recombination processes among the electrons (e), atoms (a), ions (i), and photons (ν) occur, on the average, at the same rate. For any set of particle states, the chemical reaction equation of the plasma is

$$\nu + a \rightleftarrows i + e \quad . \qquad (1)$$

Since the entropy of the particle system assumes an extremum in thermal equilibrium, the affinity of the reaction (1) vanishes. As the chemical potential of the photons (ν) is zero, the reactive equilibrium condition becomes

$$\sum_{s=a,i,e} \nu_s \mu_s = 0, \qquad \mu_\nu = 0 \quad ;$$

$$\nu_a = -1, \qquad \nu_e = +1, \qquad \nu_i = +1 \quad . \qquad (2)$$

The chemical potentials and the stoichiometric coefficients of the particle components (s) are designated by μ_s and ν_s, respectively.

According to Fermi statistics, the chemical potential μ_e of an ideal electron gas of density n_e and temperature T is given, implicitly, by the functional relation[7]

$$n_e = 2(2\pi m_e kT/h^2)^{3/2} \, U_{1/2}(\mu_e/kT) \qquad (3)$$

where

$$U_\rho(\alpha) = \frac{1}{\Gamma(\rho+1)} \int_0^\infty \frac{u^\rho du}{e^{u-\alpha}+1} \quad , \ \rho > -1 \quad . \qquad (4)$$

The Sommerfeld integrals $U_\rho(\alpha)$, $\rho = 1/2$ and $\rho = 3/2$, and their ratio $U_{3/2}(\alpha)/U_{1/2}(\alpha)$ are tabulated elsewhere.[11] Their graphical representation versus $\alpha \equiv \mu_e/kT$ is given in Fig. 2. For a classical electron gas, $\exp(-\mu_e/kT) \gg 1$, and Eqs. (3)-(4) yield explicitly

$$\mu_e^o = -kT \ln[2(2\pi m_e kT/h^2)^{3/2} n_e^{-1}] \qquad .$$

For any degree of degeneracy, the chemical potential μ_e of the electron gas can, therefore, formally be written as

$$\mu_e = -\gamma(\alpha)\, kT \ln[2(2\pi m_e kT/h^2)^{3/2} n_e^{-1}] \tag{5}$$

where

$$\gamma(\alpha) = \alpha/\ln U_{1/2}(\alpha) \ , \quad \alpha \equiv \mu_e/kT \qquad . \tag{6}$$

The factor $\gamma(\alpha)$ gives the deviation of the quantum statistical potential μ_e in Eq. (3) from its classical value μ_e^o.

In the derivation of the chemical potentials μ_s of the classical atom (s=a) and ion (s=i) components, internal electronic states σ=1,2,3,..., σ_∞ with excitation energy ε_σ^s and statistical weight g_σ^s have to be considered, in addition to the translational states of these particles. The resulting chemical potentials of the nondegenerate ion and atom components are given by[12]

$$\mu_s = -kT\{\ln[(2\pi m_s kT/h^2)^{3/2} n_s^{-1}] + \ln Z_s - (\hat{\varepsilon}_s/kT)\}, \ s=i,a \tag{7}$$

where

$$Z_s = \sum_{\sigma=1}^{\sigma_\infty} g_\sigma^s\, e^{-\varepsilon_\sigma^s/kT} \ , \quad g_\sigma^s = 2J_\sigma^s + 1, \quad J_\sigma^s \equiv L_\sigma^s + S_\sigma^s \qquad , \tag{8}$$

are the excitational sums of states, and $\hat{\varepsilon}_\sigma^s$ are the (internal) ground state energies of the ions (s=i) and atoms (s=a), respectively. L_σ^s designates the orbital and S_σ^s the spin quantum numbers.

Substitution of μ_s, s=i,a, from Eq. (7) and μ_e from Eq. (5) into Eq. (2) yields the following fundamental equation, which determines the ionization equilibrium in plasma with electrons of arbitrary degree of degeneracy:

$$\frac{n_i n_e^{\gamma(\alpha)}}{n_a} = \frac{Z_i}{Z_a} 2^{\gamma(\alpha)} \left(\frac{2\pi m_e kT}{h^2}\right)^{\frac{3}{2}\gamma(\alpha)} e^{-\varepsilon/kT} \qquad (9)$$

where

$$\varepsilon = \hat{\varepsilon}_i - \hat{\varepsilon}_a > 0 \qquad (10)$$

is the ionization energy of the atoms (a). The power $\gamma(\alpha)$ in Eq. (9) is according to Eq. (6) a function of n_e and T, since α is a function of n_e and T by Eq. (3),

$$\gamma = \alpha/\ln U_{1/2}(\alpha),\ \alpha = \tilde{U}_{1/2}\left(n_e/2(2\pi m_e kT/h^2)^{3/2}\right) \qquad (11)$$

where $\tilde{U}_{1/2}(x)$ is the inverse function to the integral functional $U_{1/2}(\alpha) = x$ defined by Eq. (4).

A complete system of equations for the particle densities n_e, n_i, and n_a is obtained by supplementing Eqs. (9) and (11) with the equation of electrical neutrality (12) and the equation (13) for the plasma pressure P:

$$n_e = n_i \qquad , \qquad (12)$$

$$P = n_a kT + n_i kT + p_e,\quad p_e = 2(2\pi m_e kT/h^2)^{3/2} U_{3/2}(\alpha)\, kT \qquad . \qquad (13)$$

In accordance with the plasma model, $p_s = n_s kT$ for s=i,a whereas p_e is given by the quantum statistical equation of state.[7] The equations (9), (11), (12), and (13) represent four independent equations, which determine the unknowns n_e, n_i, n_a, and $\gamma(\alpha)$ for given values of P and T.

The above thermal ionization theory is valid up to a critical pressure P_{cr}, at which the wave functions of the outer electrons of the atoms begin to overlap.[13] For $P > P_{cr}$, all atoms are at least singly ionized due to quantum mechanical pressure ionization.[13] In view of the success of ideal Fermi statistics for electrons in metals ($n_e \approx 10^{24}$ cm^{-3}), it is probably reasonable to neglect nonideal effects on

the equation of state (13) as a first approximation. On the other hand, the lowering $\Delta\varepsilon$ of the ionization energy ε and the suppression of the higher excited states of the atoms and ions ($\sigma_\infty < \infty$) by nonideal effects are significant even at moderate electron densities, and have to be considered since $Z_s = \infty$ for $\sigma \to \infty$ or $\Delta\varepsilon = 0$.

Since only a theory of weakly nonideal plasmas exists, we determine $\Delta\varepsilon$ from dimensional considerations for nonideal high pressure plasmas. Excluding the parameter T (which is typical for weakly nonideal systems, $\Delta\varepsilon \ll kT$), we form from the remaining plasma parameters e and n_e the combination of the dimension energy,

$$\Delta\varepsilon \cong \delta e^2 n_e^{1/3} \quad , \quad O[\delta] = 1 \quad , \tag{14}$$

which determines the order of magnitude of the lowering of the ionization energy due to Coulomb interactions. The dimensionless factor δ of order 1 is to be determined from the respective atomic model, e.g., $\delta = Z/2$ for hydrogen-like atoms of charge number Z, since the binding energy of the outer electron is $E_n = Ze^2/2a_n$ in the n-th level with the major axis of the elliptic orbit a_n. Accordingly, the ionization energy in Eqs. (9)-(10) becomes

$$\varepsilon = \varepsilon_o - \Delta\varepsilon \quad , \quad \varepsilon_o \equiv (\hat{\varepsilon}_i - \hat{\varepsilon}_a)_o \quad , \tag{15}$$

where ε_o is the hypothetical energy which is required to transform an isolated atom in ground state into an *isolated* ion in ground state. For any $\Delta\varepsilon > 0$, only a finite number ($\sigma_\infty < \infty$) of internal atomic states exist, where the integer σ_∞ is given by

$$\sigma_\infty = \text{Integer } \sigma \left(\varepsilon^a_{\sigma=\infty} - \varepsilon^a_\sigma \quad = \Delta\varepsilon\right) \quad . \tag{16}$$

For hydrogen-like atoms the excitation energies ε^a_σ are theoretically known ($\sigma \to n$), whereas for more complex atoms ε^a_σ has to be obtained from

spectroscopic tables.[14] In analogous way, σ_∞ is estimated for the ions (s=i).

In limiting cases, the quantum statistical ionization equation (9) reduces to well-known formulas. In the classical limit, $\alpha = \mu_e/kT \rightarrow -\infty$:

$$\frac{n_i n_e}{n_a} = 2\,\frac{Z_i}{Z_a}\left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} e^{-\varepsilon/kT} \quad , \quad \gamma(\alpha) \rightarrow 1, \tag{17}$$

we obtain the Saha equation. In the intermediate limit, $U_{1/2}(\alpha) \rightarrow 1 \pm 0$ or $\gamma(\alpha) \rightarrow \pm\,\infty$:

$$n_e = 2\left(2\pi m_e kT/h^2\right)^{3/2} \quad , \quad |\gamma(\alpha)| \rightarrow +\,\infty \quad , \tag{18}$$

we obtain the critical density $\tilde{n}_e$ of the partially degenerate electron gas. This result is in accord with Fig. 2.

The limiting cases indicate that the strongest deviations from the Saha equation (17) occur if the crucial parameter $|\gamma(\alpha)|$ is large. Fig. 3 shows $\gamma(\alpha)$ versus α based on Eqs. (6) and (4). It is seen that $\gamma(\alpha)$ deviates from its classical value, $\gamma(\alpha) = 1$ for $-\alpha << -1$, in the region $-1 < \alpha < \infty$. It is $\gamma(\alpha) = 0$ for $\alpha = 0$, then $\gamma(\alpha)$ decreases to $\gamma = -\infty$ and jumps to $\gamma = +\,\infty$ already within the region $0 < \alpha < 0.5$. After reaching a minimum $\gamma \cong 2$ at $\alpha \cong 2$, $\gamma(\alpha)$ increases without limit for further increasing α, $\gamma(\alpha) \rightarrow \infty$ for $\alpha \rightarrow \infty$.

APPLICATIONS

In applications of the above theory to the calculation of the composition of high pressure plasmas, a major difficulty results from the impossibility to eliminate α from the Sommerfeld integral functionals by analytical methods. For this reason n_e, n_i, and n_a are eliminated from Eqs. (3), (9) and (12)-(13) to obtain an equation for α:

$$[U_{1/2}(\alpha)]^{\gamma(\alpha)+1} = n_a \frac{Z_i}{Z_a} 2^{-1}(2\pi m_e kT/h^2)^{-3/2} e^{-\varepsilon/kT} \quad , \tag{19a}$$

$$n_a \equiv P/kT - 2(2\pi m_e kT/h^2)^{3/2}[U_{1/2}(\alpha) + U_{3/2}(\alpha)] \quad . \tag{19b}$$

The integral functional $\gamma(\alpha)$ is defined in Eq. (6). For given (fixed) values of P and T, α is determined from Eq. (19) by numerical iteration. In the lowest approximation, $\alpha = \alpha_o$ is computed from Eq. (19) for $\Delta\varepsilon = \Delta\varepsilon_o = 0$ and $Z_i/Z_a = g_1^i/g_1^a$, and $n_e = n_{eo}$ from Eq. (3) for $\alpha = \alpha_o$. In the next approximation, $\Delta\varepsilon > 0$, $\sigma_\infty < \infty$, and $Z_{i,a}(\sigma_\infty)$ are calculated from Eqs. (14), (16) and (8) for the previous value $n_e = n_{eo}$. This process is repeated until a convergent set of values α, n_e, $\Delta\varepsilon$, and σ_∞ is obtained. Then, n_a is calculated by substituting the final α-value into Eq. (19b).

a) <u>Cs-Plasma.</u> In Fig. 4, n_e and n_a of a cesium plasma are plotted versus P = 1 -10^4 Bar for T = 10^4 °K based on the quantum-statistical ionization theory (————) and compared with the corresponding values calculated from the Saha equation (- - - -). It is seen that n_e is somewhat smaller and n_a somewhat larger than the classical values at any pressure P. The depression of the ionization due to partial degeneracy increases with increasing P. Note that the curves in Fig. 4 are physically

meaningful only up to a pressure $P \cong 10^3$ Bar since the wave functions of the Cs atoms begin to overlap at a density $n_a \cong 10^{21}cm^{-3}$.

b) H-Plasma. In Fig. 5, n_e and n_a of a hydrogen plasma are plotted versus $P = 1 - 10^6$ Bar for $T = 6 \times 10^4$ °K based on the quantum statistical ionization theory (————) and compared with the corresponding values calculated from the Saha equation (-------). Again n_e(Fermi) < n_e(Saha), and vice versa for n_a, at any value of P. The depression of ionization, however, is less pronounced than in the case of Cs. The wave functions of the H-atoms begin to overlap at a density $n_a \cong 10^{24}cm^{-3}$ corresponding to a pressure $P \cong 10^6$ Bar if $T = 6 \times 10^4$ °K.

In the examples a) and b) molecular states of Cs and H were disregarded in view of the high temperatures $T = 10^4$ °K and $T = 6 \times 10^4$ °K, respectively. Both examples demonstrate that the Saha equation permits to calculate in some approximation the thermal ionization of plasmas at all pressures for which the wave functions of the atoms do not overlap. The quantum statistical ionization theory is required only for refined composition calculations in this pressure range. The depression of the ionization is apparently due to the decrease in the number of large energy states in the Fermi distribution (compared to the Boltzmann distribution), which are in statistical equilibrium with the bound electron states. It is remarkable that the Saha equation agrees approximately with the quantum statistical ionization equation at pressures where the electrons are already partially to considerably degenerate.

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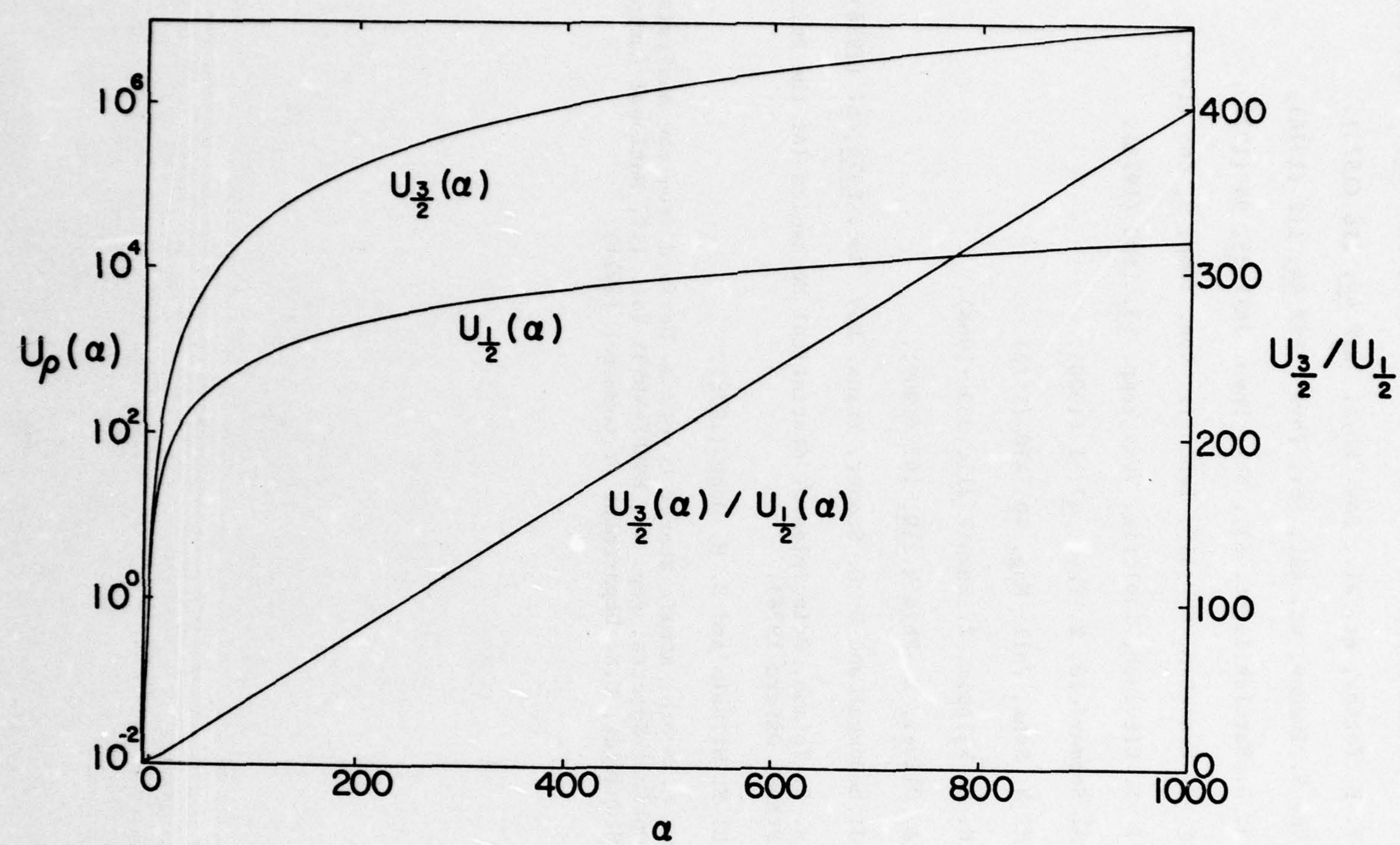

Fig. 2: $U_{1/2}(\alpha)$, $U_{3/2}(\alpha)$, and $U_{3/2}(\alpha)/U_{1/2}(\alpha)$ versus $\alpha \equiv \mu_e/kT$.

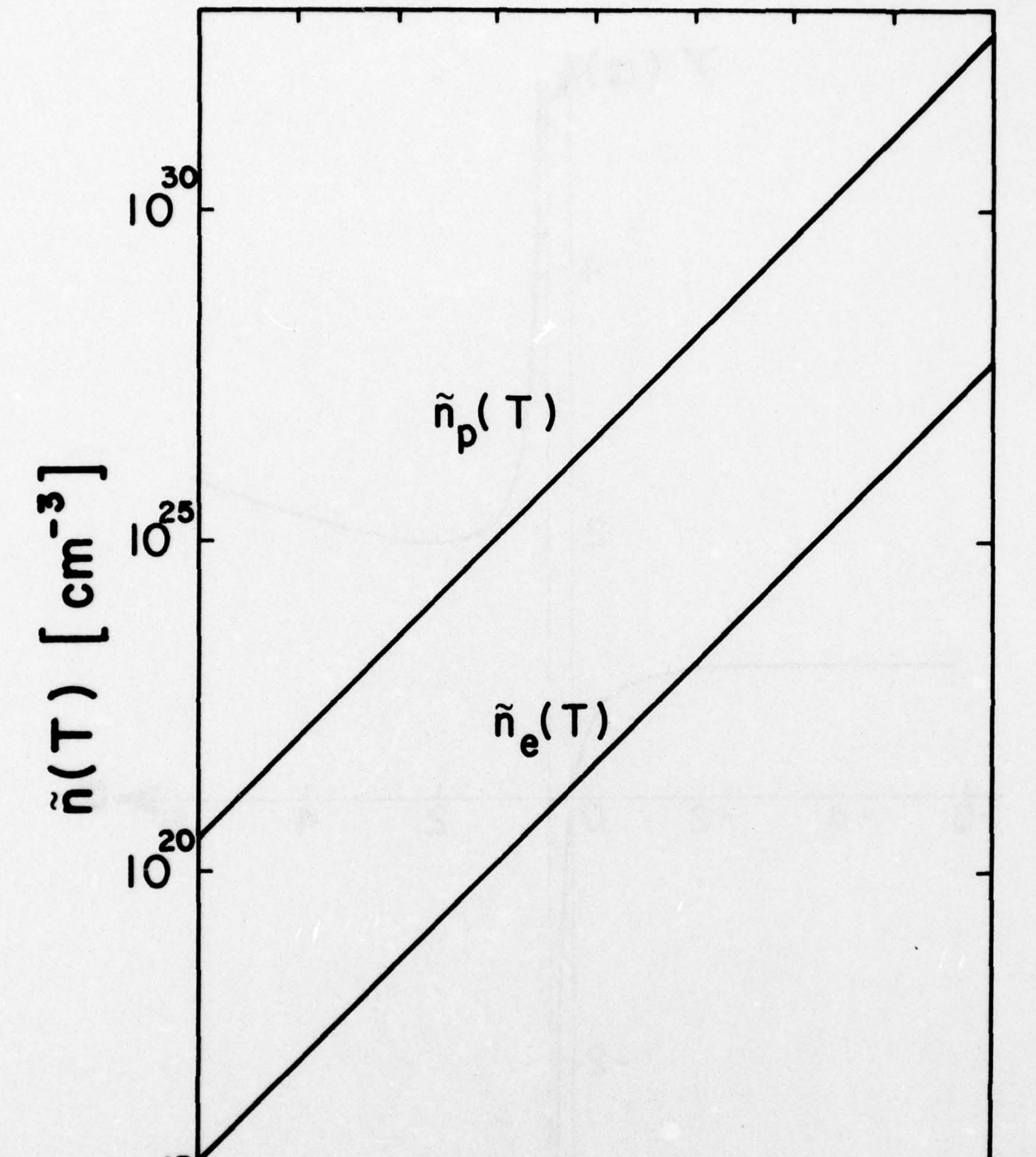

Fig. 1: Critical density $\tilde{n}$ versus T for electrons (e) and protons (p).

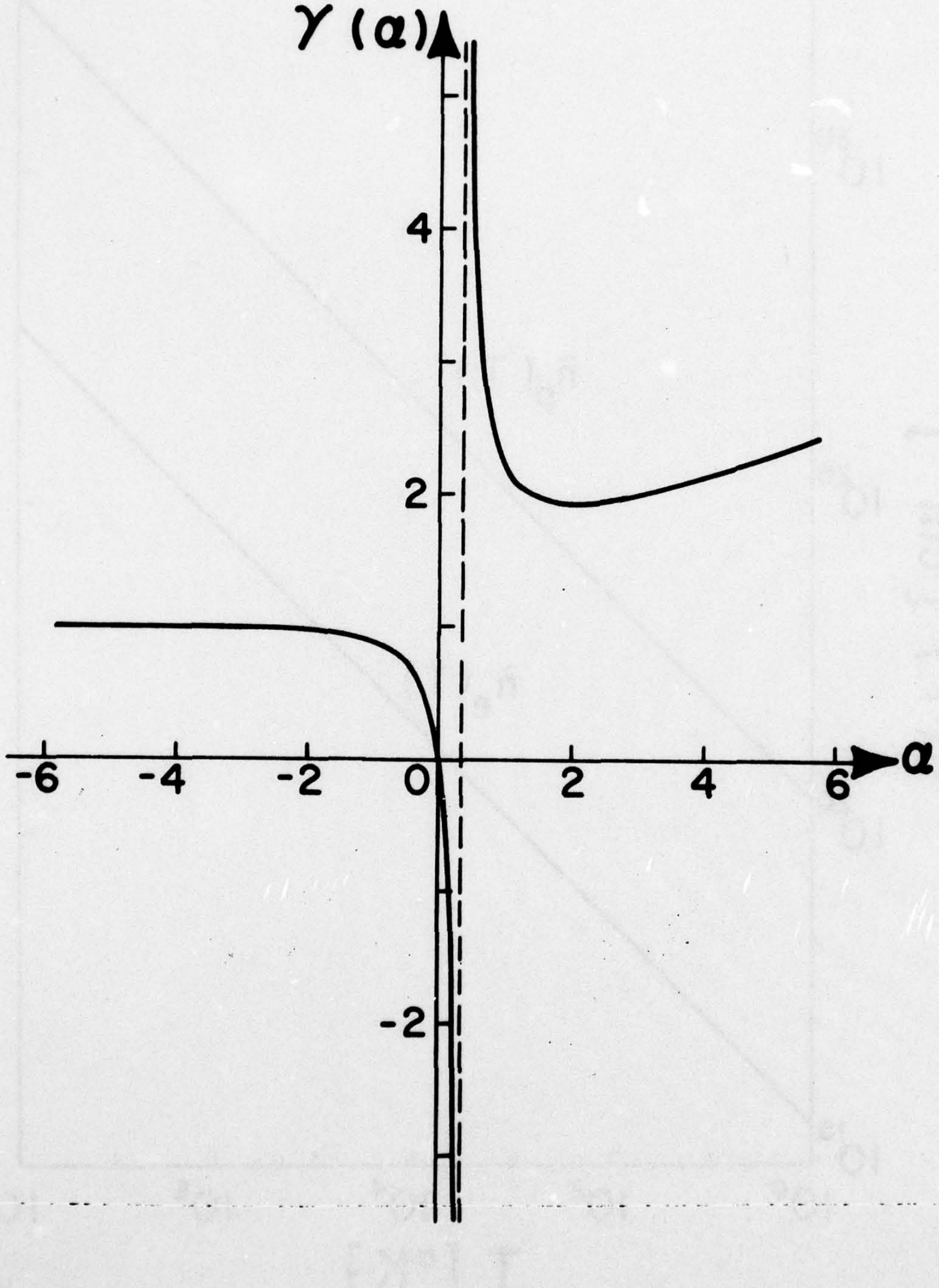

Fig. 3: $\gamma(\alpha) \equiv \alpha/\ln U_{1/2}(\alpha)$ versus $\alpha \equiv \mu_e/kT$.

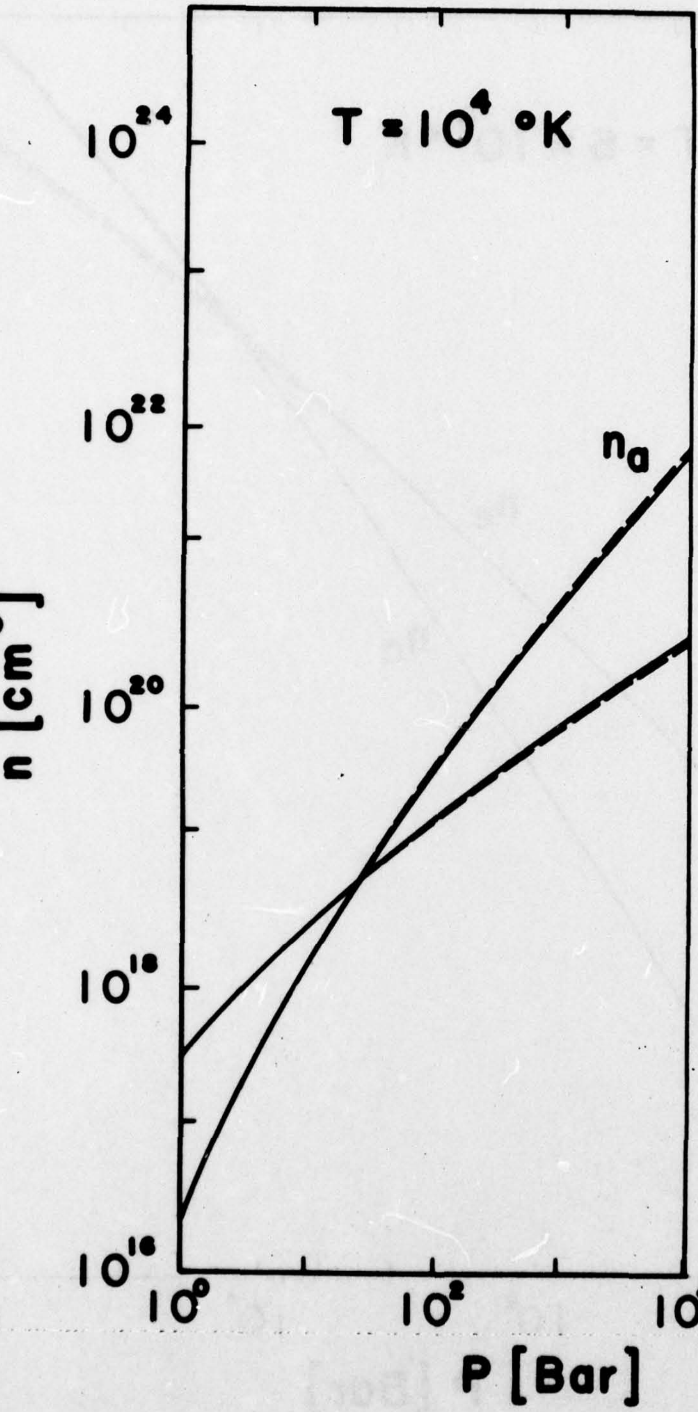

Fig. 4: n_e and n_a of Cs-plasma versus P for T = 10^4 ^oK according to quantum statistical (——) and Saha (---) ionization equations.

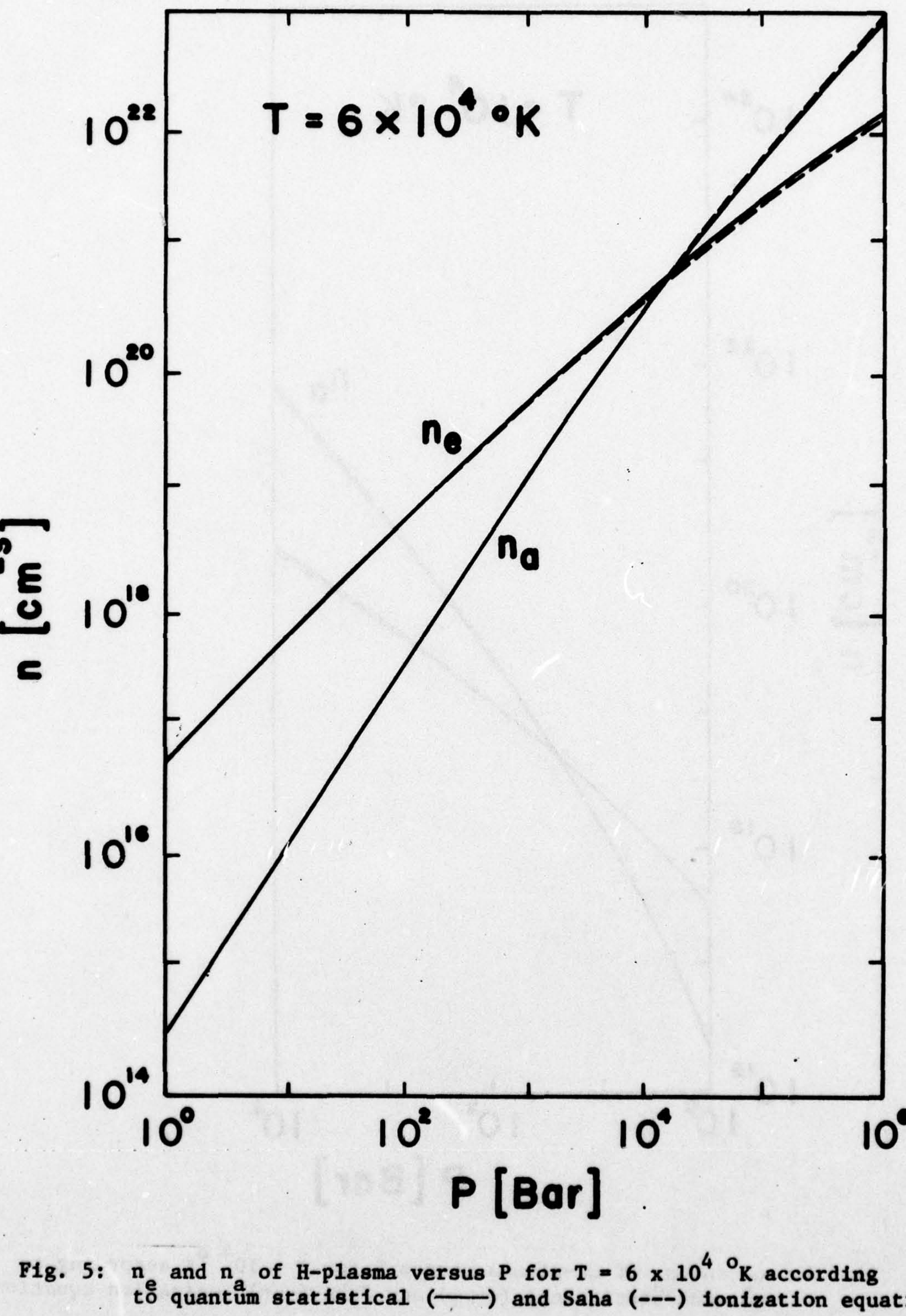

Fig. 5: n_e and n_a of H-plasma versus P for T = 6 x 10^4 °K according to quantum statistical (——) and Saha (---) ionization equations.

V. INTEGRAL TABLES FOR THERMODYNAMIC FUNCTIONS OF FERMION SYSTEMS*

Abstract

The energy density e, pressure p, entropy density s, and chemical potential μ of ideal Fermion systems are related to the Sommerfeld integrals $U_{1/2}(\alpha)$ and $U_{3/2}(\alpha)$, respectively, where $\alpha \equiv \mu/kT$. For the quantitative evaluation of these thermodynamic functions, the integrals $U_{1/2}(\alpha)$, $U_{3/2}(\alpha)$, and the ratio $U_{3/2}(\alpha)/U_{1/2}(\alpha)$ are tabulated for the interval $-5 \leq \alpha \leq 1000$ which covers the cases of nondegenerate, partially degenerate, and degenerate Fermion systems.

I. INTRODUCTION

Fermions are particles which have a spin $I = \hbar s$, $s = 1/2, 3/2, 5/2, \ldots$, the statistics of which is, therefore, subject to the Pauli exclusion principle according to which at most one Fermion can occupy a quantum state defined by a complete set of quantum numbers including the spin quantum number [1)]. The wave function of a system of identical Fermions is antisymmetric against the exchange of the coordinates of two arbitrary particles of the system[2)]. The quantum statistics of Fermion systems is important in astrophysics[3)], solid state physics[4)], atomic[5)] and nuclear[6)] physics. As examples, it is referred to the structure theory of the white dwarfs[7)], the theory of the transport properties of metals[8)], the statistical theory of many-electron atoms[9)], and the Fermi gas model of the nucleus[10)]. Fermi statistics also forms the basis of quantum plasma physics[11)]. Recently, plasmas have been generated at pressures up to 10^7 atm by means of shockwave compression, which exhibit quantum effects such as degeneracy, exchange forces, and metal-like transport behavior[12)].

Taking into consideration that the occupation number of a quantum state for Fermions is either 0 or 1, Fermi derived the nonrelativistic distribution functin $f(\varepsilon)$ for the translational energy ε of an ideal system of similar Fermions (without internal degrees of freedom) in the form[1)]:

$$f(\varepsilon) = \frac{\rho(\varepsilon)}{e^{(\varepsilon - \mu)/kT} + 1}, \quad 0 \leq \varepsilon < \infty, \tag{1}$$

where

$$\rho(\varepsilon) = g_s\, 2\pi(2m/h^2)^{3/2}\varepsilon^{1/2}, \quad g_s = 2s + 1, \tag{2}$$

is the density of states at the energy ε, g_s is the statistical weight of the Fermion, and T is the absolute temperature in °K (m = particle mass,

h = Planck constant, k = Boltzmann constant). The chemical potential μ per particle is determined by the zero-order (ε^0) moment of $f(\varepsilon)$[1],

$$n = 2\pi g_s (2m/h^2)^{3/2} \int_0^\infty \frac{\varepsilon^{1/2} d\varepsilon}{e^{(\varepsilon - \mu)/kT} + 1} > 0 \; . \tag{3}$$

Eq. (3) gives $\mu = \mu(n,T)$ implicitly (through an integral functional) as a function of density n and temperature T of the particles. According to Eq. (1), $f(\varepsilon) = \rho(\varepsilon)/2$ for $\varepsilon = \mu$, $f(\varepsilon) \cong \rho(\varepsilon)$ for $\varepsilon \ll \mu$, and $f(\varepsilon) \cong 0$ for $\varepsilon \gg \mu$. The particle system is degenerate for $A \ll 1$, non-degenerate for $A \gg 1$, and partially degenerate for $A \cong 1$, where

$$A = e^{-\mu/kT} > 0 \quad . \tag{4}$$

For $A \gg 1$ or $-\mu/kT \gg 1$ ($\mu < 0$), Eqs. (1) and (3) reduce to the energy distribution and chemical potential of the Boltzmann gas,

$$f(\varepsilon) \cong g_s 2\pi (2m/h^2)^{3/2} \varepsilon^{1/2} e^{(\mu - \varepsilon)/kT}, \; A \gg 1, \tag{5}$$

$$\mu \cong -kT \ln[g_s (2\pi m kT/h^2)^{3/2} n^{-1}], \; A \gg 1. \tag{6}$$

In the classical case, μ is given explicitly as a function of n and T by Eq. (6). From the condition A = 1, the critical density $\tilde{n}$, which separates nondegenerate ($n \ll \tilde{n}$) and degenerate ($n \gg \tilde{n}$) systems at a given temperature T, is obtained[13],

$$\tilde{n} = g_s (2\pi m kT/h^2)^{3/2} \quad . \tag{7}$$

Fig. 1 exhibits $\tilde{n}$ versus T for electrons ($g_s = 2$, $m = 9.1096 \times 10^{-28}$ gr) and protons ($g_s = 2$, $m = 1.6726 \times 10^{-24}$ gr).

The thermodynamic functions of Fermion systems contain to the so-called Sommerfeld integrals[13] $U_\rho(\mu/kT)$, $\rho = 1/2, 3/2$, which can not be evaluated in closed form. In view of the general importance of the thermodynamic functions of Fermion systems[3-12], the integrals $U_{1/2}(\mu/kT)$, $U_{3/2}(\mu/kT)$, and the ratio $U_{3/2}(\mu/kT)/U_{1/2}(\mu/kT)$ are tabulated herein for $\mu/kT = -5$ to 1000. The ratio $U_{3/2}(\mu/kT)/U_{1/2}(\mu/kT)$ represents the modification of the thermodynamic functions due to Fermi statistics.

II. THERMODYNAMIC FUNCTIONS

For Fermion systems, the energy density e is given as the first order (ϵ^1) moment of the distribution function (1). The pressure p is given in terms of e by the virial equation, $p = 2e/3$. The entropy density s is given by the thermodynamic relation $sT = e + p - n\mu$. Thus, the thermodynamic functions of ideal systems of Fermions (without internal degrees of freedom) are obtained as[1,13]:

$$e = g_s \frac{3}{2} kT(2\pi mkT/h^2)^{3/2} U_{3/2}(\mu/kT) \ , \tag{8}$$

$$p = g_s kT(2\pi mkT/h^2)^{3/2} U_{3/2}(\mu/kT) \ , \tag{9}$$

$$s = g_s k (2\pi mkT/h^2)^{3/2} [\frac{5}{2} U_{3/2}(\mu/kT) - U_{1/2}(\mu/kT) \frac{\mu}{kT}] \ , \tag{10}$$

where

$$n = g_s (2\pi mkT/h^2)^{3/2} U_{1/2}(\mu/kT) \tag{11}$$

and

$$U_\rho(\mu/kT) = \frac{1}{\Gamma(\rho + 1)} \int_0^\infty \frac{u^\rho du}{e^{u - (\mu/kT)} + 1} \ , \rho = 1/2, 3/2. \tag{12}$$

Eq. (11) determines the chemical potential μ implicitly as a function of n and T. In view of this relation, $\mu = \mu(p,T)$, Eq. (8) - (10) relate e, p, and s to n and T. Eq. (12) defines the Sommerfeld integrals[13].

By combining Eq. (11) with Eqs. (8) - (10), the thermodynamic functions are obtained in the form:

$$e = \frac{3}{2} nkT \frac{U_{3/2}(\mu/kT)}{U_{1/2}(\mu/kT)} , \tag{13}$$

$$p = nkT \frac{U_{3/2}(\mu/kT)}{U_{1/2}(\mu/kT)} , \tag{14}$$

$$s = nk[\frac{5}{2} \frac{U_{3/2}(\mu/kT)}{U_{1/2}(\mu/kT)} - \frac{\mu}{kT}] , \tag{15}$$

where

$$\frac{\mu}{kT} = U_{1/2}^{-1}(n/g_s(2\pi mkT/h^2)^{3/2}) \tag{16}$$

in accordance with Eq. (11) if $U_{1/2}^{-1}$ designates the inverse function to the functional $U_{1/2}$. Equations (13) - (15) indicate that the thermodynamic functions of Fermion systems differ from the corresponding classical formulae by the factor $U_{3/2}(\mu/kT)/U_{1/2}(\mu/kT)$.

Fig. 2 gives an overview over the tabulated functions $U_{1/2}(\alpha)$, $U_{3/2}(\alpha)$, and $U_{3/2}(\alpha)/U_{1/2}(\alpha)$ for $-5 \leq \alpha \leq 1000$, $\alpha = \mu/kT$. $U_{1/2}(\alpha)$ and $U_{3/2}(\alpha)$ are positive, monotonically increasing functions of α in the interval $-\infty < \alpha < \infty$. It is $U_{3/2}(\alpha) > U_{1/2}(\alpha)$, and $U_{3/2}(\alpha)/U_{1/2}(\alpha) \propto \alpha$ for large arguments, $\alpha >> 1$.

III. EVALUATION OF U_ρ-INTEGRALS

For the numerical evaluation of the integral functionals, $U_\rho(\mu/kT)$ is rewritten in the forms

$$U_\rho(\alpha) = \frac{1}{\Gamma(\rho+1)} \int_0^\infty \frac{x^\rho dx}{e^{x-\alpha}+1} \ , \ \rho = \frac{1}{2}, \frac{3}{2} \qquad , \qquad (17)$$

where

$$\alpha \equiv \mu/kT \quad , \quad -\infty < \alpha < +\infty \qquad . \qquad (18)$$

It is $\alpha << 0$ ($A >> 1$) for nondegenerate systems and $\alpha >> 1$ ($A << 1$) for degenerate systems, and $\alpha \cong 0$ ($A \cong 1$) in the transition region. Eq. (17) satisfies the differential relation,

$$dU_\rho(\alpha)/d\alpha = [\rho\Gamma(\rho)/\Gamma(\rho+1)]U_{\rho-1}(\alpha), \quad \rho > 0. \qquad (19)$$

Before describing the numerical procedure on which the following integral tables are based, a brief review of the series expansions for $U_\rho(\alpha)$ is presented.

A. Series Expansions

For negative arguments, $\alpha < 0$, $U_\rho(\alpha)$ can be approximated by the semiconvergent expansion:[13)]

$$U_\rho(\alpha) = \lim_{n\to\infty} \sum_{m=1}^{n} (-1)^{m-1}(e^\alpha)^m/m^{\rho+1} \quad , \quad \alpha < 0 \qquad . \qquad (20)$$

The number of summation terms n_α required for an accuracy of 1 in the 7-th decimal place of $U_\rho(\alpha)$ is given in Scheme 1.

For large arguments, $\alpha >> 1$, $U_\rho(\alpha)$ can be approximated by the non-convergent series expansion:[13)]

$$U_\rho(\alpha) \cong \frac{\alpha^{\rho+1}}{\Gamma(\rho+2)} [1+2 \sum_{m=1}^{n_\alpha} {}_{\rho+1}P_{2m} \ \eta(2m)/\alpha^{2m}], \ \alpha >> 1 \qquad , \qquad (21)$$

where

$$ {}_{\rho+1}P_{2m} = (\rho+1)\rho(\rho-1)\cdots(\rho+2-2m) \quad , \tag{22} $$

$$ \eta(2m) = \sum_{\nu=1}^{\infty} (-1)^{\nu-1}/\nu^{2m} \quad . \tag{23} $$

As n becomes large, $n > n_\alpha$, Eq. (21) becomes inaccurate especially for $0 \le \alpha \le 10$, and finally blows up in the limit $n\to\infty$ due to factorial increase of ${}_{\rho+1}P_{2m}$. The optimum number n_α of summation terms which provides the best approximation to $U_\rho(\alpha)$ is given in Scheme 2.

B. Numerical Integration

In view of the convergence difficulties of the expansion in Eq. (21) and the lack of a suitable expansion for $0 \le \alpha \le 10$, a systematic numerical method is used to integrate $U_\rho(\alpha)$ in Eq. (17) for the region $-5 < \alpha < 1000$ by means of a computer. The numerical integration of $U_\rho(\alpha)$ is based on the integration formula[15)]

$$ \int_0^\infty \frac{x^\rho dx}{e^{x-\alpha}+1} = \lim_{n\to\infty} \sum_{m=1}^{n} \int_{(m-1)d}^{md} \frac{x^\rho dx}{e^{x-\alpha}+1} \equiv \sum_{m=1}^{n} I_m + R_n \quad . \tag{24} $$

In Eq. (24), d is an increment of the argument values x, and R_n is the remainder if only $n < \infty$ terms are summed. The Gaussian quadrature rule[15)] is used to compute the integral values I_m, which are compared with those obtained by the trapezoidal rule in connection with Romberg's extrapolation method.[15)] In order to achieve an accuracy of at least seven significant figures for $U_\rho(\alpha)$, the upper limit to the remainder R_n is taken as

$$ R_n < 10^{-9} \times \sum_{m=1}^{n} I_m \quad . \tag{25} $$

The error in ν-point Gaussian quadrature formula for the integral,

$$ I_m = \int_{(m-1)d}^{md} f(x)dx \quad , \tag{26} $$

is[16])

$$\varepsilon_{\nu} = \frac{(\nu!)^4}{(2\nu+1)[(2\nu)!]^3} d^{2\nu+1} f^{(2\nu)}(\xi) \quad , \tag{27}$$

where ξ is some point in $((m-1)d, md)$, and the integrand $f(x)$ has a continuous derivative of order 2ν in $((m-1)d, md)$.

In order to minimize the error caused by the Gaussian quadrature formula, the evaluation is carried through with appropriate increment d for each α-value by means of a 32-point Gaussian quadrature formula, which exactly integrates a polynomial of degree 63 or less.

In the Tables (IV), the functional values of $U_\rho(\alpha)$ for $\rho = 1/2$ and 3/2 are given either to six decimal places or to seven significant figures. The intervals in the argument α are so chosen that interpolation may yield the maximum attainable accuracy over most of the range covered.

The Tables (IV) cover a range of α from -5 to +1000. The numerical values of $U_{1/2}(\alpha)$, $U_{3/2}(\alpha)$, and $U_{3/2}(\alpha)/U_{1/2}(\alpha)$ are listed for $-5 \leq \alpha \leq 0$ and $10 < \alpha \leq 100$ with an interval of 0.1, for $0 < \alpha \leq 10$ with interval 0.01, and for $100 < \alpha \leq 1000$ with an interval 10. All significant figures are regarded as accurate, except the last significant digit with possible uncertainty one.

As an illustration to the use of the Tables (IV), the chemical potential μ is determined for given values of n and T by inversion according to Eq. (16). From Eq. (11),

$$n = CT^{3/2} U_{1/2}(\alpha) \quad , \quad \alpha = \mu/kT \quad ,$$

with

$$C = 4.8291 \times 10^{15} \text{ cm}^{-3}\, {}^{o}\text{K}^{-3/2} \text{ for electrons} \quad ,$$

$$C = 3.7994 \times 10^{20} \text{ cm}^{-3}\, {}^{o}\text{K}^{-3/2} \text{ for protons} \quad .$$

Hence,

$$U_{1/2}(\alpha) = 5.8571 \times 10^{2} \text{ for electrons} \quad ,$$

$$U_{1/2}(\alpha) = 7.4444\times 10^{-3} \text{ for protons} \quad ,$$

for

$$n = 10^{24} \text{ cm}^{-3} \quad , \quad T = 5 \times 10^{3} \, {}^{o}K \quad .$$

Accordingly, by the Tables (IV),

$$\alpha \cong 84.62 \text{ or } \mu \cong 5.84 \times 10^{-11} \text{ erg for electrons} \quad ,$$

$$\alpha \cong -4.90 \text{ or } \mu \cong -3.38 \times 10^{-12} \text{ erg for protons} \quad .$$

If an accuracy to more decimal points is required, one has to interpolate between successive table values by means of one of the interopolation methods.[15,16)]

IV. INTEGRAL TABLES

The following tables give the numerical values of the Sommerfeld integrals and their ratio,

$$U_{1/2}(\alpha) \ , \qquad U_{3/2}(\alpha) \ , \qquad U_{3/2}(\alpha)/U_{1/2}(\alpha) \ ,$$

to seven significant figures or six decimal places in dependence of $\alpha = \mu/kT$ with a step size $\Delta\alpha$:

$$-5 \leq \alpha \leq 0 \ , \qquad \Delta\alpha = 10^{-1} \ ,$$
$$0 \leq \alpha \leq 10^{1} \ , \qquad \Delta\alpha = 10^{-2} \ ,$$
$$10^{1} \leq \alpha \leq 10^{2} \ , \qquad \Delta\alpha = 10^{-1} \ ,$$
$$10^{2} \leq \alpha \leq 10^{3} \ , \qquad \Delta\alpha = 10^{+1} \ .$$

α	n_α
-4	3
-3	4
-2	7
-1	13
-0.8	16
-0.6	21
-0.4	30
-0.2	56

SCHEME 1:[14)]

Sufficient number n_α of summation terms.

$\Delta\alpha$	n_α
1.314- 3.696	1
3.696- 5.776	2
5.776- 7.817	3
7.817- 9.847	4
9.847-11.87	5
11.87 -13.89	6
13.89 -15.90	7
15.90 -17.91	8
17.91 -19.92	9
19.92 -21.93	10

SCHEME 2:[14)]

Optimum number n_α of summation terms.

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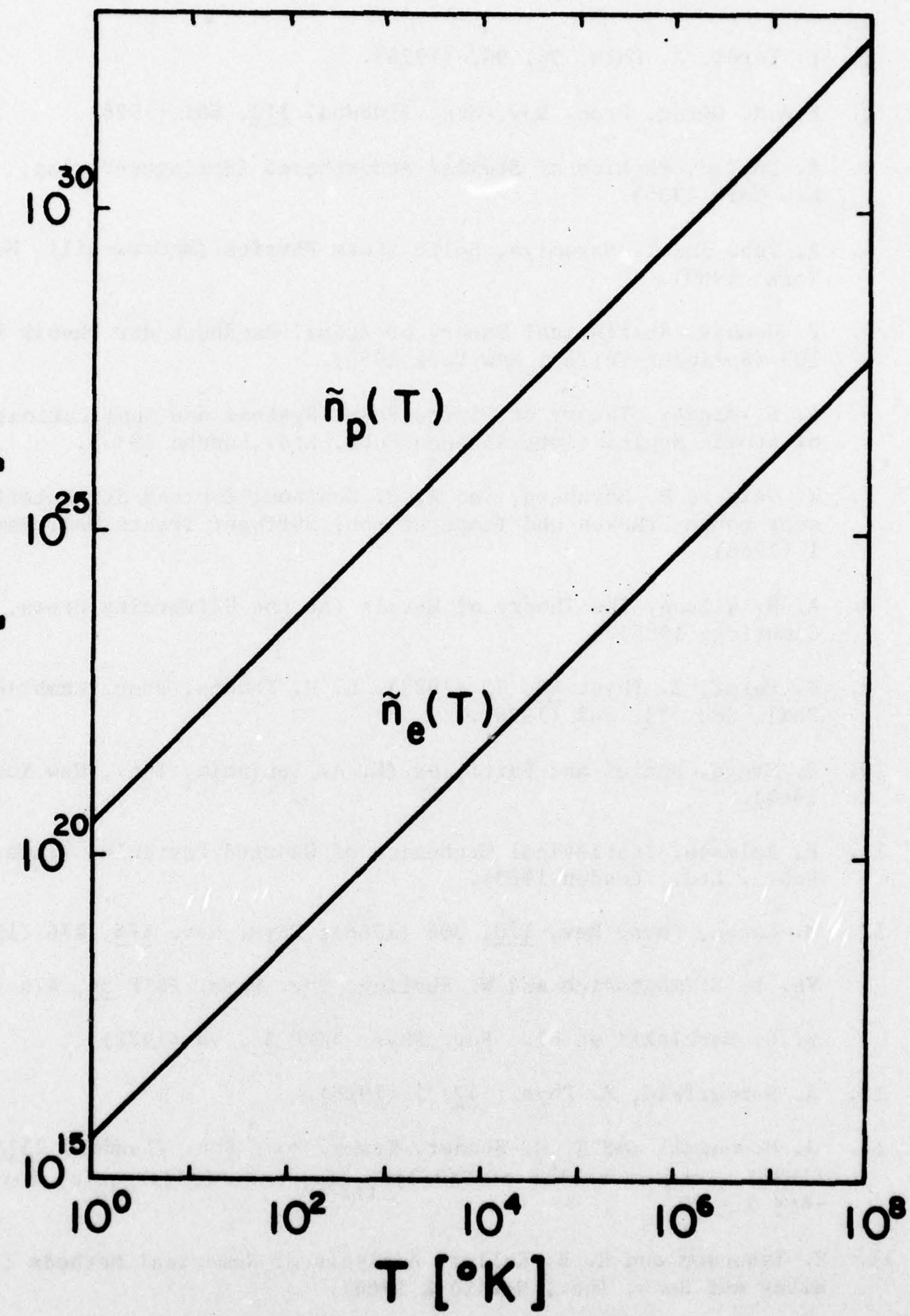

Fig. 1: Critical density ñ versus T for electrons (e) and protons (p).

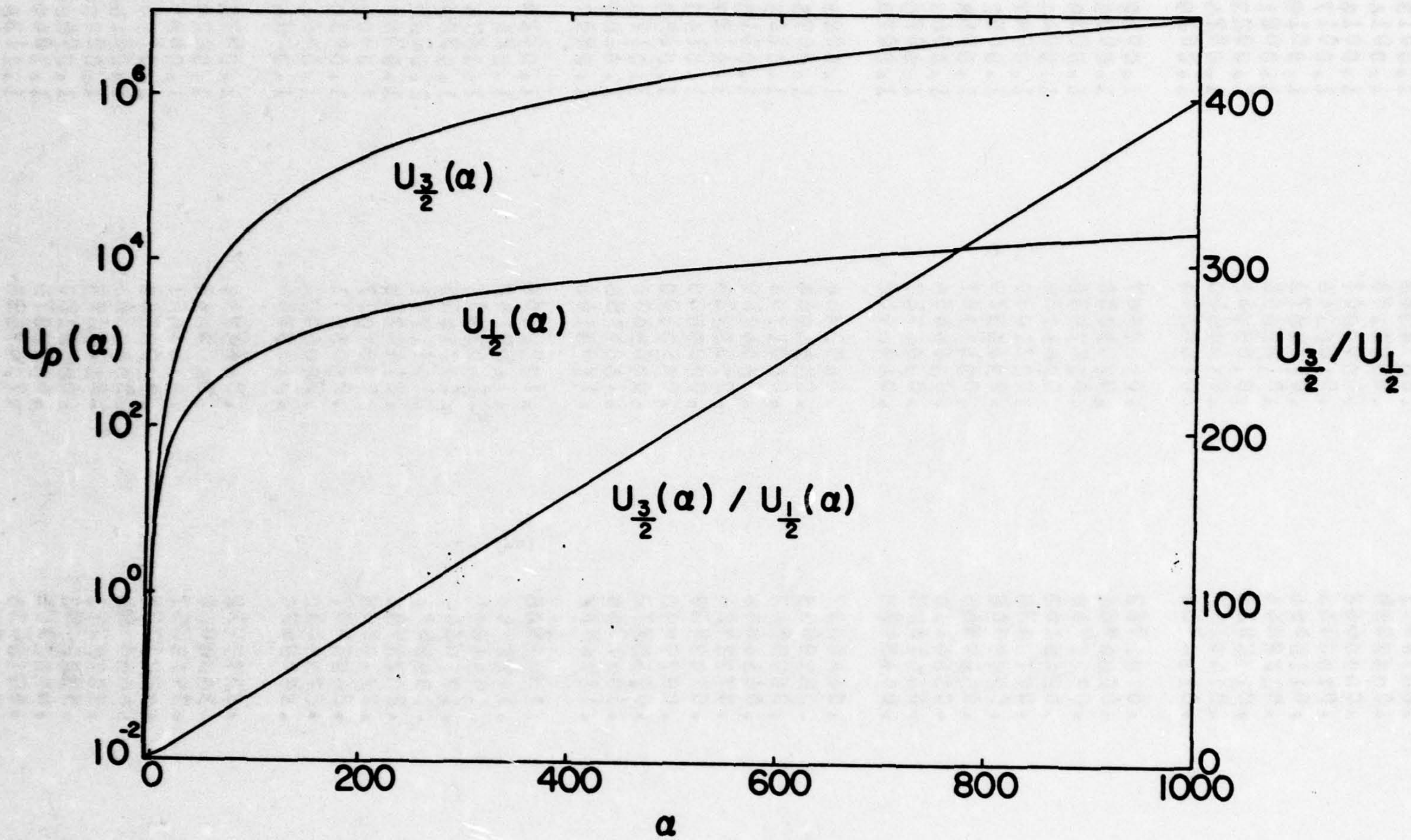

Fig. 2: $U_{1/2}(\alpha)$, $U_{3/2}(\alpha)$, and $U_{3/2}(\alpha)/U_{1/2}(\alpha)$ versus $\alpha \equiv \mu/kT$.

$\alpha = \mu/kT$	$U_{1/2}(\alpha)$	$U_{3/2}(\alpha)$	$U_{3/2}(\alpha)/U_{1/2}(\alpha)$
-5.0	.0067220	.0067299	1.001188
-4.9	.0074271	.0074368	1.001313
-4.8	.0082059	.0082178	1.001450
-4.7	.0090662	.0090807	1.001602
-4.6	.0100163	.0100340	1.001770
-4.5	.0110656	.0110873	1.001956
-4.4	.0122244	.0122508	1.002160
-4.3	.0135039	.0135362	1.002387
-4.2	.0149167	.0149560	1.002636
-4.1	.0164764	.0165244	1.002912
-4.0	.0181982	.0182567	1.003216
-3.9	.0200986	.0201700	1.003552
-3.8	.0221960	.0222830	1.003922
-3.7	.0245103	.0246164	1.004331
-3.6	.0270636	.0271930	1.004782
-3.5	.0298802	.0300379	1.005279
-3.4	.0329865	.0331787	1.005828
-3.3	.0364116	.0366458	1.006432
-3.2	.0401875	.0404727	1.007099
-3.1	.0443488	.0446962	1.007833
-3.0	.0489337	.0493566	1.008642
-2.9	.0539838	.0544984	1.009533
-2.8	.0595443	.0601704	1.010514
-2.7	.0656646	.0664259	1.011594
-2.6	.0723984	.0733238	1.012781
-2.5	.0798039	.0809280	1.014086
-2.4	.0879440	.0893090	1.015521
-2.3	.0968872	.0985436	1.017096
-2.2	.1067068	.1087156	1.018826
-2.1	.1174822	.1199168	1.020723
-2.0	.1292985	.1322468	1.022802
-1.9	.1422468	.1458142	1.025079
-1.8	.1564242	.1607371	1.027572
-1.7	.1719344	.1771435	1.030297
-1.6	.1888871	.1951721	1.033274
-1.5	.2073982	.2149728	1.036522
-1.4	.2275898	.2367077	1.040063
-1.3	.2495897	.2605511	1.043918
-1.2	.2735310	.2866903	1.048109
-1.1	.2995820	.3153266	1.052661
-1.0	.3277952	.3466748	1.057596
-.9	.3584068	.3809645	1.062939
-.8	.3915357	.4184401	1.068715
-.7	.4273327	.4593606	1.074948
-.6	.4659495	.5040006	1.081664
-.5	.5075371	.5526495	1.088885
-.4	.5522453	.6056120	1.096636
-.3	.6002206	.6632075	1.104940
-.2	.6516056	.7257698	1.113818
-.1	.7065376	.7936468	1.123290
0.0	.7651471	.8671999	1.133377

$\alpha = \mu/kT$	$U_{1/2}(\alpha)$	$U_{3/2}(\alpha)$	$U_{3/2}(\alpha)/U_{1/2}(\alpha)$
.01	.771215	.874882	1.134420
.02	.777321	.882624	1.135469
.03	.783466	.890478	1.136525
.04	.789648	.898294	1.137587
.05	.795870	.906221	1.138656
.06	.802129	.914211	1.139730
.07	.808428	.922264	1.140812
.08	.814765	.930380	1.141899
.09	.821142	.938559	1.142993
.10	.827557	.946803	1.144094
.11	.834011	.955111	1.145201
.12	.840505	.963483	1.146315
.13	.847038	.971921	1.147435
.14	.853611	.980424	1.148561
.15	.860223	.988993	1.149694
.16	.866875	.997629	1.150834
.17	.873566	1.006331	1.151980
.18	.880298	1.015100	1.153133
.19	.887069	1.023937	1.154292
.20	.893881	1.032842	1.155458
.21	.900733	1.041815	1.156630
.22	.907625	1.050857	1.157809
.23	.914558	1.059967	1.158995
.24	.921530	1.069148	1.160187
.25	.928544	1.078398	1.161386
.26	.935598	1.087719	1.162592
.27	.942693	1.097110	1.163804
.28	.949829	1.106573	1.165023
.29	.957006	1.116107	1.166249
.30	.964224	1.125713	1.167481
.31	.971483	1.135392	1.168720
.32	.978783	1.145143	1.169966
.33	.986125	1.154967	1.171218
.34	.993508	1.164866	1.172477
.35	1.000932	1.174838	1.173743
.36	1.008398	1.184884	1.175016
.37	1.015906	1.195006	1.176296
.38	1.023456	1.205203	1.177582
.39	1.031047	1.215475	1.178875
.40	1.038680	1.225824	1.180175
.41	1.046355	1.236249	1.181481
.42	1.054072	1.246751	1.182795
.43	1.061831	1.257330	1.184115
.44	1.069633	1.267988	1.185442
.45	1.077476	1.278723	1.186776
.46	1.085363	1.289537	1.188117
.47	1.093291	1.300431	1.189464
.48	1.101262	1.311403	1.190819
.49	1.109275	1.322456	1.192180
.50	1.117332	1.333589	1.193548

$\alpha = \mu/kT$	$U_{1/2}(\alpha)$	$U_{3/2}(\alpha)$	$U_{3/2}(\alpha)/U_{1/2}(\alpha)$
.51	1.125430	1.344803	1.194923
.52	1.133572	1.356098	1.196305
.53	1.141756	1.367474	1.197694
.54	1.149984	1.378933	1.199089
.55	1.158254	1.390474	1.200492
.56	1.166567	1.402098	1.201901
.57	1.174923	1.413806	1.203317
.58	1.183323	1.425597	1.204740
.59	1.191765	1.437472	1.206171
.60	1.200251	1.449432	1.207607
.61	1.208780	1.461477	1.209051
.62	1.217353	1.473608	1.210502
.63	1.225968	1.485825	1.211960
.64	1.234628	1.498127	1.213424
.65	1.243331	1.510517	1.214896
.66	1.252077	1.522994	1.216374
.67	1.260867	1.535559	1.217860
.68	1.269700	1.548212	1.219352
.69	1.278578	1.560953	1.220851
.70	1.287499	1.573783	1.222357
.71	1.296463	1.586703	1.223870
.72	1.305472	1.599713	1.225390
.73	1.314524	1.612813	1.226917
.74	1.323621	1.626003	1.228451
.75	1.332761	1.639285	1.229992
.76	1.341945	1.652659	1.231540
.77	1.351174	1.666124	1.233094
.78	1.360446	1.679682	1.234656
.79	1.369762	1.693333	1.236224
.80	1.379123	1.707078	1.237800
.81	1.388528	1.720916	1.239382
.82	1.397977	1.734849	1.240971
.83	1.407470	1.748876	1.242567
.84	1.417007	1.762908	1.244170
.85	1.426589	1.777216	1.245780
.86	1.436215	1.791530	1.247397
.87	1.445885	1.805940	1.249021
.88	1.455599	1.820448	1.250652
.89	1.465358	1.835053	1.252289
.90	1.475162	1.849755	1.253934
.91	1.485009	1.864556	1.255585
.92	1.494902	1.879455	1.257244
.93	1.504838	1.894454	1.258909
.94	1.514819	1.909552	1.260581
.95	1.524845	1.924751	1.262260
.96	1.534915	1.940049	1.263946
.97	1.545030	1.955449	1.265638
.98	1.555189	1.970950	1.267338
.99	1.565393	1.986553	1.269045
1.00	1.575641	2.002258	1.270758

$\alpha = \mu/kT$	$U_{1/2}(\alpha)$	$U_{3/2}(\alpha)$	$U_{3/2}(\alpha)/U_{1/2}(\alpha)$
1.01	1.585934	2.018066	1.272478
1.02	1.596271	2.033977	1.274205
1.03	1.606653	2.049992	1.275939
1.04	1.617080	2.066110	1.277680
1.05	1.627551	2.082333	1.279427
1.06	1.638067	2.098661	1.281162
1.07	1.648628	2.115095	1.282943
1.08	1.659233	2.131634	1.284711
1.09	1.669882	2.148280	1.286486
1.10	1.680577	2.165032	1.288267
1.11	1.691316	2.181891	1.290056
1.12	1.702099	2.198858	1.291851
1.13	1.712928	2.215933	1.293653
1.14	1.723800	2.233117	1.295461
1.15	1.734718	2.250410	1.297277
1.16	1.745680	2.267811	1.299099
1.17	1.756687	2.285323	1.300928
1.18	1.767738	2.302945	1.302764
1.19	1.778834	2.320678	1.304606
1.20	1.789975	2.338522	1.306456
1.21	1.801160	2.356478	1.308311
1.22	1.812389	2.374546	1.310174
1.23	1.823664	2.392726	1.312043
1.24	1.834983	2.411019	1.313919
1.25	1.846346	2.429426	1.315802
1.26	1.857754	2.447946	1.317691
1.27	1.869206	2.466581	1.319587
1.28	1.880703	2.485330	1.321490
1.29	1.892245	2.504195	1.323399
1.30	1.903831	2.523175	1.325315
1.31	1.915461	2.542272	1.327237
1.32	1.927136	2.561485	1.329166
1.33	1.938855	2.580815	1.331102
1.34	1.950619	2.600262	1.333044
1.35	1.962427	2.619827	1.334993
1.36	1.974280	2.639511	1.336949
1.37	1.986177	2.659313	1.338911
1.38	1.998118	2.679234	1.340879
1.39	2.010103	2.699275	1.342854
1.40	2.022133	2.719437	1.344836
1.41	2.034207	2.739718	1.346824
1.42	2.046325	2.760121	1.348818
1.43	2.058488	2.780645	1.350819
1.44	2.070695	2.801291	1.352827
1.45	2.082945	2.822059	1.354840
1.46	2.095240	2.842950	1.356861
1.47	2.107580	2.863964	1.358888
1.48	2.119963	2.885101	1.360921
1.49	2.132390	2.906363	1.362960
1.50	2.144861	2.927749	1.365006

$\alpha = \mu/kT$	$U_{1/2}(\alpha)$	$U_{3/2}(\alpha)$	$U_{3/2}(\alpha)/U_{1/2}(\alpha)$
1.51	2.157376	2.949261	1.367059
1.52	2.169936	2.970897	1.369118
1.53	2.182539	2.992659	1.371183
1.54	2.195186	3.014548	1.373254
1.55	2.207877	3.036563	1.375332
1.56	2.220611	3.058706	1.377416
1.57	2.233390	3.080976	1.379506
1.58	2.246212	3.103374	1.381603
1.59	2.259078	3.125900	1.383706
1.60	2.271988	3.148555	1.385815
1.61	2.284941	3.171340	1.387931
1.62	2.297938	3.194254	1.390053
1.63	2.310978	3.217299	1.392180
1.64	2.324062	3.240474	1.394315
1.65	2.337190	3.263780	1.396455
1.66	2.350361	3.287218	1.398601
1.67	2.363575	3.310788	1.400754
1.68	2.376833	3.334490	1.402913
1.69	2.390134	3.358324	1.405078
1.70	2.403478	3.382292	1.407249
1.71	2.416866	3.406304	1.409426
1.72	2.430297	3.430630	1.411609
1.73	2.443771	3.455000	1.413799
1.74	2.457288	3.479505	1.415994
1.75	2.470849	3.504146	1.418195
1.76	2.484452	3.528922	1.420403
1.77	2.498098	3.553835	1.422616
1.78	2.511787	3.578885	1.424836
1.79	2.525520	3.604071	1.427061
1.80	2.539295	3.629395	1.429293
1.81	2.553112	3.654857	1.431530
1.82	2.566973	3.680457	1.433773
1.83	2.580876	3.706197	1.436023
1.84	2.594822	3.732075	1.438278
1.85	2.608811	3.758093	1.440539
1.86	2.622842	3.784251	1.442806
1.87	2.636916	3.810550	1.445078
1.88	2.651032	3.836990	1.447357
1.89	2.665191	3.863571	1.449642
1.90	2.679392	3.890294	1.451932
1.91	2.693635	3.917159	1.454228
1.92	2.707921	3.944167	1.456530
1.93	2.722249	3.971318	1.458837
1.94	2.736619	3.998612	1.461151
1.95	2.751031	4.026050	1.463470
1.96	2.765485	4.053633	1.465794
1.97	2.779981	4.081360	1.468125
1.98	2.794519	4.109232	1.470461
1.99	2.809100	4.137250	1.472803
2.00	2.823722	4.165414	1.475151

$\alpha = \mu/kT$	$U_{1/2}(\alpha)$	$U_{3/2}(\alpha)$	$U_{3/2}(\alpha)/U_{1/2}(\alpha)$
2.01	2.838386	4.193725	1.477504
2.02	2.853091	4.222182	1.479862
2.03	2.867838	4.250787	1.482227
2.04	2.882627	4.279539	1.484597
2.05	2.897458	4.308440	1.486972
2.06	2.912330	4.337488	1.489353
2.07	2.927244	4.366686	1.491740
2.08	2.942199	4.396033	1.494132
2.09	2.957195	4.425530	1.496530
2.10	2.972233	4.455178	1.498933
2.11	2.987312	4.484975	1.501341
2.12	3.002432	4.514924	1.503755
2.13	3.017594	4.545024	1.506175
2.14	3.032796	4.575276	1.508600
2.15	3.048040	4.605680	1.511030
2.16	3.063324	4.636237	1.513466
2.17	3.078650	4.666947	1.515907
2.18	3.094016	4.697810	1.518353
2.19	3.109423	4.728827	1.520805
2.20	3.124871	4.759999	1.523262
2.21	3.140360	4.791325	1.525725
2.22	3.155889	4.822806	1.528192
2.23	3.171459	4.854443	1.530665
2.24	3.187069	4.886235	1.533144
2.25	3.202720	4.918184	1.535627
2.26	3.218412	4.950290	1.538116
2.27	3.234143	4.982552	1.540610
2.28	3.249915	5.014973	1.543109
2.29	3.265728	5.047551	1.545613
2.30	3.281580	5.080287	1.548122
2.31	3.297473	5.113183	1.550637
2.32	3.313405	5.146237	1.553157
2.33	3.329378	5.179451	1.555681
2.34	3.345390	5.212825	1.558211
2.35	3.361443	5.246359	1.560746
2.36	3.377535	5.280054	1.563286
2.37	3.393667	5.313910	1.565831
2.38	3.409839	5.347927	1.568381
2.39	3.426050	5.382106	1.570936
2.40	3.442301	5.416448	1.573496
2.41	3.458592	5.450953	1.576061
2.42	3.474922	5.485620	1.578631
2.43	3.491291	5.520451	1.581206
2.44	3.507700	5.555446	1.583786
2.45	3.524148	5.590605	1.586371
2.46	3.540635	5.625929	1.588960
2.47	3.557162	5.661418	1.591555
2.48	3.573728	5.697073	1.594154
2.49	3.590332	5.732893	1.596758
2.50	3.606976	5.768879	1.599367

$\alpha = \mu/kT$	$U_{1/2}(\alpha)$	$U_{3/2}(\alpha)$	$U_{3/2}(\alpha)/U_{1/2}(\alpha)$
2.51	3.623658	5.805032	1.601981
2.52	3.640380	5.841353	1.604600
2.53	3.657140	5.877840	1.607223
2.54	3.673939	5.914496	1.609851
2.55	3.690777	5.951319	1.612484
2.56	3.707653	5.988311	1.615122
2.57	3.724568	6.025472	1.617764
2.58	3.741522	6.062803	1.620411
2.59	3.758514	6.100303	1.623063
2.60	3.775544	6.137973	1.625719
2.61	3.792612	6.175814	1.628380
2.62	3.809719	6.213825	1.631045
2.63	3.826864	6.252008	1.633716
2.64	3.844048	6.290343	1.636390
2.65	3.861269	6.328889	1.639070
2.66	3.878528	6.367588	1.641754
2.67	3.895826	6.406460	1.644442
2.68	3.913161	6.445505	1.647135
2.69	3.930534	6.484723	1.649833
2.70	3.947945	6.524116	1.652535
2.71	3.965394	6.563682	1.655241
2.72	3.982880	6.603424	1.657952
2.73	4.000404	6.643340	1.660667
2.74	4.017965	6.683432	1.663387
2.75	4.035564	6.723700	1.666111
2.76	4.053201	6.764143	1.668840
2.77	4.070874	6.804764	1.671573
2.78	4.088585	6.845561	1.674310
2.79	4.106334	6.886535	1.677052
2.80	4.124119	6.927688	1.679798
2.81	4.141942	6.969018	1.682548
2.82	4.159802	7.010527	1.685303
2.83	4.177698	7.052214	1.688062
2.84	4.195632	7.094081	1.690825
2.85	4.213603	7.136127	1.693593
2.86	4.231610	7.178353	1.696364
2.87	4.249654	7.220759	1.699140
2.88	4.267735	7.263346	1.701921
2.89	4.285853	7.306114	1.704705
2.90	4.304007	7.349063	1.707493
2.91	4.322197	7.392194	1.710286
2.92	4.340425	7.435507	1.713083
2.93	4.358688	7.479003	1.715884
2.94	4.376988	7.522681	1.718689
2.95	4.395324	7.566543	1.721498
2.96	4.413697	7.610588	1.724311
2.97	4.432105	7.654817	1.727129
2.98	4.450550	7.699230	1.729950
2.99	4.469031	7.743828	1.732776
3.00	4.487548	7.788611	1.735605

$\alpha = \mu/kT$	$U_{1/2}(\alpha)$	$U_{3/2}(\alpha)$	$U_{3/2}(\alpha)/U_{1/2}(\alpha)$
3.01	4.506101	7.833579	1.738438
3.02	4.524689	7.878793	1.741276
3.03	4.543314	7.924073	1.744117
3.04	4.561974	7.969599	1.746963
3.05	4.580670	8.015312	1.749812
3.06	4.599402	8.061213	1.752665
3.07	4.618169	8.107301	1.755523
3.08	4.636972	8.153576	1.758384
3.09	4.655811	8.200040	1.761249
3.10	4.674684	8.246693	1.764118
3.11	4.693594	8.293534	1.766990
3.12	4.712538	8.340565	1.769867
3.13	4.731518	8.387785	1.772747
3.14	4.750533	8.435195	1.775631
3.15	4.769583	8.482796	1.778519
3.16	4.788668	8.530587	1.781411
3.17	4.807788	8.578569	1.784307
3.18	4.826943	8.626743	1.787206
3.19	4.846133	8.675108	1.790109
3.20	4.865358	8.723665	1.793016
3.21	4.884618	8.772415	1.795927
3.22	4.903913	8.821358	1.798841
3.23	4.923242	8.870494	1.801759
3.24	4.942606	8.919823	1.804680
3.25	4.962004	8.969346	1.807605
3.26	4.981437	9.019063	1.810534
3.27	5.000905	9.068975	1.813467
3.28	5.020406	9.119081	1.816403
3.29	5.039943	9.169383	1.819343
3.30	5.059513	9.219880	1.822286
3.31	5.079118	9.270573	1.825233
3.32	5.098757	9.321463	1.828183
3.33	5.118430	9.372549	1.831137
3.34	5.138137	9.423831	1.834095
3.35	5.157879	9.475311	1.837056
3.36	5.177654	9.526989	1.840020
3.37	5.197463	9.578865	1.842989
3.38	5.217306	9.630938	1.845960
3.39	5.237183	9.683211	1.848935
3.40	5.257093	9.735682	1.851913
3.41	5.277038	9.788353	1.854895
3.42	5.297016	9.841223	1.857881
3.43	5.317027	9.894293	1.860869
3.44	5.337072	9.947564	1.863861
3.45	5.357151	10.001035	1.866857
3.46	5.377263	10.054707	1.869856
3.47	5.397409	10.108580	1.872858
3.48	5.417588	10.162655	1.875864
3.49	5.437800	10.216932	1.878872
3.50	5.458045	10.271411	1.881885

$\alpha = \mu/kT$	$U_{1/2}(\alpha)$	$U_{3/2}(\alpha)$	$U_{3/2}(\alpha)/U_{1/2}(\alpha)$
3.51	5.478323	10.326093	1.884900
3.52	5.498635	10.380978	1.887919
3.53	5.518989	[illegible]	[illegible]
3.54	5.539376	10.491367	1.893966
3.55	5.559768	10.546853	1.896995
3.56	5.580212	10.602553	1.900027
3.57	5.600689	10.658457	1.903062
3.58	5.621198	10.714567	1.906100
3.59	5.641740	10.770881	1.909142
3.60	5.662315	10.827402	1.912186
3.61	5.682922	10.884128	1.915234
3.62	5.703562	10.941060	1.918285
3.63	5.724235	10.998199	1.921340
3.64	5.744940	11.055545	1.924397
3.65	5.765678	11.113098	1.927457
3.66	5.786448	11.170859	1.930521
3.67	5.807250	11.228827	1.933588
3.68	5.828085	11.287004	1.936657
3.69	5.848952	11.345389	1.939730
3.70	5.869851	11.403983	1.942806
3.71	5.890783	11.462786	1.945885
3.72	5.911746	11.521799	1.948967
3.73	5.932742	11.581021	1.952052
3.74	5.953770	11.640454	1.955140
3.75	5.974829	11.700096	1.958231
3.76	5.995921	11.759950	1.961325
3.77	6.017044	11.820015	1.964422
3.78	6.038199	11.880291	1.967522
3.79	6.059386	11.940779	1.970625
3.80	6.080605	12.001479	1.973731
3.81	6.101855	12.062391	1.976840
3.82	6.123137	12.123516	1.979952
3.83	6.144451	12.184854	1.983066
3.84	6.165796	12.246405	1.986184
3.85	6.187172	12.308170	1.989305
3.86	6.208581	12.370149	1.992428
3.87	6.230020	12.432342	1.995554
3.88	6.251491	12.494749	1.998683
3.89	6.272993	12.557372	2.001815
3.90	6.294526	12.620209	2.004950
3.91	6.316091	12.683262	2.008087
3.92	6.337686	12.746531	2.011228
3.93	6.359313	12.810016	2.014371
3.94	6.380971	12.873718	2.017517
3.95	6.402660	12.937636	2.020666
3.96	6.424380	13.001771	2.023817
3.97	6.446131	13.066123	2.026972
3.98	6.467912	13.130694	2.030129
3.99	6.489725	13.195482	2.033288
4.00	6.511568	13.260488	2.036451

$\alpha = \mu/kT$	$U_{1/2}(\alpha)$	$U_{3/2}(\alpha)$	$U_{3/2}(\alpha)/U_{1/2}(\alpha)$
4.01	6.533442	13.325713	2.039616
4.02	6.555347	1[illegible].391157	2.042784
4.03	6.577282	1[illegible].456820	2.045954
4.04	6.59924[illegible]	[illegible].522703	2.049128
4.05	6.621[illegible]	[illegible].588805	2.052304
4.06	6.64[illegible]	[illegible].655128	2.055482
4.07	6.[illegible]30	[illegible].721671	2.058663
4.08	6.[illegible]418	1[illegible].788434	2.061847
4.09	6.[illegible]09536	1[illegible]855419	2.065034
4.10	[illegible]731685	13.[illegible]2625	2.068223
4.11	6.753864	13.99[illegible]53	2.071415
4.12	6.776073	14.05[illegible]3	2.074609
4.13	6.798313	14.125[illegible]	2.077806
4.1[illegible]	6.820583	14.1936[illegible]	2.081005
[illegible]	6.842883	14.261986	2.084207
[illegible]	6.865213	14.330527	2.087412
[illegible]17	6.887573	14.399201	2.090619
[illegible].18	6.909962	14.468278	2.093829
4.19	6.932382	14.537400	2.097041
4.20	6.954832	14.606996	2.100256
4.21	6.977312	14.676587	2.103473
4.22	6.999822	14.746472	2.106693
4.23	7.022361	14.816583	2.109915
4.24	7.044930	14.886990	2.113139
4.25	7.067529	14.957482	2.116367
4.26	7.090157	15.028270	[illegible].119596
4.27	7.112815	15.099285	[illegible]1226[illegible]
4.28	7.135503	15.170527	[illegible]260[illegible]
4.29	7.158220	15.241995	2.[illegible]93[illegible]
4.30	7.180967	15.313601	2.[illegible]53
	7.203743	15.385615	2.13[illegible]1
	[illegible]	15.457766	2.139[illegible]
		15.530146	2.1422[illegible]
		15.[illegible]2754	2.14552[illegible]
			2.148771
			2.152025

$\alpha = \mu/kT$	$U_{1/2}(\alpha)$	$U_{3/2}(\alpha)$	$U_{3/2}(\alpha)/U_{1/2}(\alpha)$
4.01	6.533442	13.325713	2.039616
4.02	6.555347	13.391157	2.042784
4.03	6.577282	13.456820	2.045954
4.04	6.599248	13.522703	2.049128
4.05	6.621245	13.588805	2.052304
4.06	6.643272	13.655128	2.055482
4.07	6.665330	13.721671	2.058663
4.08	6.687418	13.788434	2.061847
4.09	6.709536	13.855419	2.065034
4.10	6.731685	13.922625	2.068223
4.11	6.753864	13.990053	2.071415
4.12	6.776073	14.057703	2.074609
4.13	6.798313	14.125575	2.077806
4.14	6.820583	14.193669	2.081005
4.15	6.842883	14.261986	2.084207
4.16	6.865213	14.330527	2.087412
4.17	6.887573	14.399291	2.090619
4.18	6.909962	14.468278	2.093829
4.19	6.932382	14.537490	2.097041
4.20	6.954832	14.606926	2.100256
4.21	6.977312	14.676587	2.103473
4.22	6.999822	14.746472	2.106693
4.23	7.022361	14.816583	2.109915
4.24	7.044930	14.886920	2.113139
4.25	7.067529	14.957482	2.116367
4.26	7.090157	15.028270	2.119596
4.27	7.112815	15.099285	2.122828
4.28	7.135503	15.170527	2.126063
4.29	7.158220	15.241995	2.129300
4.30	7.180967	15.313691	2.132539
4.31	7.203743	15.385615	2.135781
4.32	7.226548	15.457766	2.139025
4.33	7.249383	15.530146	2.142271
4.34	7.272248	15.602754	2.145520
4.35	7.295141	15.675591	2.148771
4.36	7.318064	15.748657	2.152025
4.37	7.341016	15.821952	2.155281
4.38	7.363997	15.895477	2.158539
4.39	7.387007	15.969232	2.161800
4.40	7.410047	16.043217	2.165063
4.41	7.433115	16.117433	2.168328
4.42	7.456213	16.191880	2.171596
4.43	7.479339	16.266558	2.174866
4.44	7.502494	16.341467	2.178138
4.45	7.525679	16.416608	2.181412
4.46	7.548892	16.491980	2.184689
4.47	7.572134	16.567585	2.187968
4.48	7.595404	16.643423	2.191249
4.49	7.618704	16.719494	2.194533
4.50	7.642032	16.795797	2.197818

$\alpha = \mu/kT$	$U_{1/2}(\alpha)$	$U_{3/2}(\alpha)$	$U_{3/2}(\alpha)/U_{1/2}(\alpha)$
4.51	7.665389	16.872334	2.201106
4.52	7.688774	16.949105	2.204396
4.53	7.712188	17.026110	2.207689
4.54	7.735631	17.103349	2.210983
4.55	7.759102	17.180623	2.214280
4.56	7.782601	17.258531	2.217579
4.57	7.806129	17.336475	2.220880
4.58	7.829685	17.414654	2.224183
4.59	7.853270	17.493069	2.227488
4.60	7.876883	17.571719	2.230796
4.61	7.900524	17.650606	2.234106
4.62	7.924194	17.729730	2.237417
4.63	7.947892	17.809090	2.240731
4.64	7.971618	17.888688	2.244047
4.65	7.995372	17.968523	2.247366
4.66	8.019154	18.048505	2.250686
4.67	8.042964	18.128906	2.254008
4.68	8.066802	18.209455	2.257333
4.69	8.090668	18.290242	2.260659
4.70	8.114562	18.371268	2.263987
4.71	8.138485	18.452533	2.267318
4.72	8.162435	18.534038	2.270651
4.73	8.186412	18.615782	2.273985
4.74	8.210418	18.697766	2.277322
4.75	8.234451	18.779991	2.280661
4.76	8.258512	18.862455	2.284002
4.77	8.282601	18.945161	2.287344
4.78	8.306718	19.028107	2.290689
4.79	8.330862	19.111295	2.294036
4.80	8.355033	19.194725	2.297385
4.81	8.379233	19.278396	2.300735
4.82	8.403459	19.362309	2.304088
4.83	8.427714	19.446465	2.307443
4.84	8.451996	19.530864	2.310799
4.85	8.476305	19.615505	2.314158
4.86	8.500641	19.700390	2.317518
4.87	8.525005	19.785518	2.320881
4.88	8.549396	19.870890	2.324245
4.89	8.573815	19.956506	2.327611
4.90	8.598261	20.042366	2.330979
4.91	8.622734	20.128471	2.334349
4.92	8.647234	20.214821	2.337721
4.93	8.671761	20.301416	2.341095
4.94	8.696315	20.388257	2.344471
4.95	8.720897	20.475343	2.347848
4.96	8.745505	20.562675	2.351228
4.97	8.770141	20.650253	2.354609
4.98	8.794804	20.738078	2.357992
4.99	8.819493	20.826149	2.361377
5.00	8.844210	20.914467	2.364764

$\alpha = \mu/kT$	$U_{1/2}(\alpha)$	$U_{3/2}(\alpha)$	$U_{3/2}(\alpha)/U_{1/2}(\alpha)$
5.01	8.868953	21.003033	2.368153
5.02	[illegible]	21.091646	2.371543
5.03	[illegible]	[illegible]	2.374935
5.04	8.943344	[illegible]	2.378329
5.05	8.968194	21.359775	2.381725
5.06	8.993072	21.449581	2.385123
5.07	9.017976	21.539636	2.388522
5.08	9.042906	21.629941	2.391924
5.09	9.067863	21.720494	2.395327
5.10	9.092847	21.811298	2.398731
5.11	9.117858	21.902351	2.402138
5.12	9.142895	21.993655	2.405546
5.13	9.167958	22.085209	2.408956
5.14	9.193048	22.177014	2.412368
5.15	9.218164	22.269070	2.415781
5.16	9.243307	22.361378	2.419197
5.17	9.268476	22.453937	2.422614
5.18	9.293672	22.546747	2.426032
5.19	9.318893	22.639810	2.429453
5.20	9.344141	22.733125	2.432875
5.21	9.369416	22.826693	2.436298
5.22	9.394716	22.920514	2.439724
5.23	9.420043	23.014587	2.443151
5.24	9.445396	23.108915	2.446580
5.25	9.470775	23.203495	2.450010
5.26	9.496180	23.298330	2.453442
5.27	9.521611	23.393419	2.456876
5.28	9.547069	23.488763	2.460311
5.29	9.572552	23.584361	2.463748
5.30	9.598061	23.680214	2.467187
5.31	9.623596	23.776322	2.470627
5.32	9.649158	23.872686	2.474069
5.33	9.674745	23.969305	2.477513
5.34	9.700357	24.066180	2.480958
5.35	9.725996	24.163312	2.484405
5.36	9.751661	24.260700	2.487853
5.37	9.777351	24.358346	2.491303
5.38	9.803067	24.456248	2.494755
5.39	9.828809	24.554407	2.498208
5.40	9.854576	24.652824	2.501662
5.41	9.880370	24.751499	2.505119
5.42	9.906188	24.850431	2.508577
5.43	9.932033	24.949622	2.512036
5.44	9.957903	25.049072	2.515497
5.45	9.983798	25.148781	2.518959
5.46	10.009719	25.248748	2.522423
5.47	10.035666	25.348975	2.525889
5.48	10.061638	25.449461	2.529356
5.49	10.087635	25.550208	2.532824
5.50	10.113658	25.651214	2.536294

$\alpha = \mu/kT$	$U_{1/2}(\alpha)$	$U_{3/2}(\alpha)$	$U_{3/2}(\alpha)/U_{1/2}(\alpha)$
5.51	10.139706	25.752481	2.539766
5.52	10.165779	25.854008	2.543239
5.53	10.191876	25.955707	2.546714
5.54	10.218001	26.057846	2.550190
5.55	10.244152	26.160167	2.553667
5.56	10.270327	26.262729	2.557146
5.57	10.296527	26.365563	2.560627
5.58	10.322752	26.468660	2.564109
5.59	10.349002	26.572019	2.567592
5.60	10.375277	26.675640	2.571077
5.61	10.401578	26.779524	2.574564
5.62	10.427903	26.883671	2.578051
5.63	10.454254	26.988082	2.581541
5.64	10.480630	27.092757	2.585031
5.65	10.507030	27.197605	2.588524
5.66	10.533456	27.302897	2.592017
5.67	10.559906	27.408364	2.595512
5.68	10.586382	27.514006	2.599008
5.69	10.612882	27.620002	2.602506
5.70	10.639407	27.726353	2.606005
5.71	10.665957	27.832880	2.609506
5.72	10.692532	27.939673	2.613008
5.73	10.719132	28.046731	2.616511
5.74	10.745756	28.154055	2.620016
5.75	10.772405	28.261646	2.623522
5.76	10.799079	28.369503	2.627030
5.77	10.825778	28.477628	2.630539
5.78	10.852501	28.586019	2.634049
5.79	10.879249	28.694678	2.637561
5.80	10.906022	28.803604	2.641073
5.81	10.932819	28.912798	2.644588
5.82	10.959640	29.022261	2.648103
5.83	10.986486	29.131901	2.651620
5.84	11.013357	29.241990	2.655139
5.85	11.040252	29.352258	2.658658
5.86	11.067172	29.462705	2.662179
5.87	11.094116	29.573602	2.665702
5.88	11.121084	29.684678	2.669225
5.89	11.148077	29.796024	2.672750
5.90	11.175094	29.907639	2.676276
5.91	11.202136	30.019525	2.679804
5.92	11.229202	30.131682	2.683333
5.93	11.256292	30.244110	2.686863
5.94	11.283406	30.356808	2.690394
5.95	11.310545	30.469778	2.693927
5.96	11.337708	30.583019	2.697461
5.97	11.364895	30.696532	2.700996
5.98	11.392106	30.810317	2.704532
5.99	11.419341	30.924374	2.708070
6.00	11.446600	31.038704	2.711609

$\alpha = \mu/kT$	$U_{1/2}(\alpha)$	$U_{3/2}(\alpha)$	$U_{3/2}(\alpha)/U_{1/2}(\alpha)$
6.01	11.473884	31.153306	2.715149
6.02	11.501192	31.268182	2.718691
6.03	11.528523	31.383330	2.722233
6.04	11.555879	31.498752	2.725777
6.05	11.583259	31.614448	2.729322
6.06	11.610662	31.730417	2.732869
6.07	11.638090	31.846661	2.736417
6.08	11.665541	31.963179	2.739965
6.09	11.693017	32.079972	2.743515
6.10	11.720516	32.197040	2.747067
6.11	11.748039	32.314382	2.750619
6.12	11.775586	32.432000	2.754173
6.13	11.803157	32.549894	2.757728
6.14	11.830752	32.668064	2.761284
6.15	11.858370	32.786509	2.764841
6.16	11.886012	32.905231	2.768400
6.17	11.913678	33.024230	2.771959
6.18	11.941367	33.143505	2.775520
6.19	11.969081	33.263057	2.779082
6.20	11.996817	33.382886	2.782645
6.21	12.024578	33.502993	2.786210
6.22	12.052362	33.623378	2.789775
6.23	12.080170	33.744041	2.793342
6.24	12.108001	33.864982	2.796909
6.25	12.135856	33.986201	2.800478
6.26	12.163734	34.107699	2.804048
6.27	12.191636	34.229476	2.807620
6.28	12.219561	34.351531	2.811192
6.29	12.247510	34.473867	2.814765
6.30	12.275482	34.596482	2.818340
6.31	12.303477	34.719377	2.821916
6.32	12.331496	34.842551	2.825493
6.33	12.359539	34.966006	2.829071
6.34	12.387604	35.089742	2.832650
6.35	12.415693	35.213759	2.836230
6.36	12.443806	35.338056	2.839811
6.37	12.471941	35.462635	2.843393
6.38	12.500100	35.587495	2.846977
6.39	12.528282	35.712637	2.850561
6.40	12.556487	35.838061	2.854147
6.41	12.584716	35.963767	2.857734
6.42	12.612967	36.089755	2.861322
6.43	12.641242	36.216026	2.864910
6.44	12.669540	36.342580	2.868500
6.45	12.697861	36.469417	2.872091
6.46	12.726205	36.596537	2.875683
6.47	12.754572	36.723941	2.879277
6.48	12.782962	36.851629	2.882871
6.49	12.811376	36.979600	2.886466
6.50	12.839812	37.107856	2.890062

$\alpha = \mu/kT$	$U_{1/2}(\alpha)$	$U_{3/2}(\alpha)$	$U_{3/2}(\alpha)/U_{1/2}(\alpha)$
6.51	12.868271	37.236397	2.893660
6.52	12.896753	37.365222	2.897258
6.53	12.925259	37.494332	2.900857
6.54	12.953787	37.623727	2.904456
6.55	12.982338	37.753406	2.908059
6.56	13.010911	37.883374	2.911662
6.57	13.039508	38.013626	2.915265
6.58	13.068128	38.144164	2.918870
6.59	13.096770	38.274989	2.922475
6.60	13.125435	38.406100	2.926082
6.61	13.154123	38.537407	2.929690
6.62	13.182834	38.669182	2.933298
6.63	13.211567	38.801154	2.936908
6.64	13.240323	38.933413	2.940518
6.65	13.269102	39.065961	2.944130
6.66	13.297904	39.198796	2.947742
6.67	13.326728	39.331919	2.951356
6.68	13.355575	39.465330	2.954971
6.69	13.384444	39.599030	2.958586
6.70	13.413337	39.733019	2.962203
6.71	13.442251	39.867297	2.965820
6.72	13.471188	40.001864	2.969438
6.73	13.500148	40.136721	2.973058
6.74	13.529130	40.271867	2.976678
6.75	13.558135	40.407304	2.980299
6.76	13.587163	40.543030	2.983922
6.77	13.616212	40.679047	2.987545
6.78	13.645284	40.815354	2.991169
6.79	13.674379	40.951953	2.994794
6.80	13.703496	41.088842	2.998420
6.81	13.732636	41.226023	3.002047
6.82	13.761797	41.363495	3.005675
6.83	13.790982	41.501259	3.009304
6.84	13.820188	41.639314	3.012934
6.85	13.849417	41.777662	3.016565
6.86	13.878668	41.916303	3.020196
6.87	13.907941	42.055236	3.023829
6.88	13.937237	42.194462	3.027462
6.89	13.966555	42.333981	3.031097
6.90	13.995895	42.473703	3.034732
6.91	14.025257	42.613899	3.038368
6.92	14.054642	42.754208	3.042006
6.93	14.084048	42.894902	3.045644
6.94	14.113477	43.035979	3.049283
6.95	14.142928	43.177261	3.052922
6.96	14.172401	43.318838	3.056563
6.97	14.201896	43.460709	3.060205
6.98	14.231413	43.602876	3.063847
6.99	14.260953	43.745338	3.067491
7.00	14.290514	43.888095	3.071135

$\alpha = \mu/kT$	$U_{1/2}(\alpha)$	$U_{3/2}(\alpha)$	$U_{3/2}(\alpha)/U_{1/2}(\alpha)$
7.01	14.320097	44.031148	3.074780
7.02	14.349703	44.174497	3.078426
7.03	14.379330	44.318142	3.082073
7.04	14.408979	44.462083	3.085721
7.05	14.438650	44.606322	3.089369
7.06	14.468344	44.750857	3.093019
7.07	14.498059	44.895689	3.096669
7.08	14.527796	45.040818	3.100320
7.09	14.557554	45.186245	3.103972
7.10	14.587335	45.331969	3.107625
7.11	14.617138	45.477991	3.111279
7.12	14.646962	45.624312	3.114933
7.13	14.676808	45.770931	3.118589
7.14	14.706676	45.917848	3.122245
7.15	14.736566	46.065064	3.125902
7.16	14.766477	46.212579	3.129560
7.17	14.796411	46.360394	3.133219
7.18	14.826365	46.508508	3.136879
7.19	14.856342	46.656921	3.140539
7.20	14.886340	46.805635	3.144200
7.21	14.916360	46.954648	3.147862
7.22	14.946402	47.103962	3.151525
7.23	14.976465	47.253576	3.155189
7.24	15.006550	47.403491	3.158853
7.25	15.036657	47.553707	3.162519
7.26	15.066785	47.704224	3.166185
7.27	15.096934	47.855043	3.169852
7.28	15.127106	48.006163	3.173519
7.29	15.157298	48.157585	3.177188
7.30	15.187513	48.309309	3.180857
7.31	15.217748	48.461335	3.184527
7.32	15.248006	48.613664	3.188198
7.33	15.278284	48.766296	3.191870
7.34	15.308585	48.919230	3.195542
7.35	15.338906	49.072467	3.199216
7.36	15.369249	49.226008	3.202890
7.37	15.399614	49.379852	3.206564
7.38	15.429999	49.534000	3.210240
7.39	15.460406	49.688452	3.213916
7.40	15.490835	49.843209	3.217593
7.41	15.521285	49.998269	3.221271
7.42	15.551756	50.153634	3.224950
7.43	15.582248	50.309304	3.228629
7.44	15.612762	50.465279	3.232309
7.45	15.643297	50.621560	3.235990
7.46	15.673853	50.778145	3.239672
7.47	15.704431	50.935037	3.243355
7.48	15.735030	51.092234	3.247038
7.49	15.765650	51.249737	3.250722
7.50	15.796291	51.407547	3.254406

$\alpha = \mu/kT$	$U_{1/2}(\alpha)$	$U_{3/2}(\alpha)$	$U_{3/2}(\alpha)/U_{1/2}(\alpha)$
7.51	[illegible]	[illegible]	3.258092
7.52	15.857646	[illegible]	3.261778
7.53	15.888347	51.882816	3.265465
7.54	15.919067	52.041863	3.269152
7.55	15.949814	52.201167	3.272841
7.56	15.980581	52.360849	3.276530
7.57	16.011370	52.520809	3.280219
7.58	16.042181	52.681077	3.283910
7.59	16.073012	52.841653	3.287601
7.60	16.103864	53.002537	3.291293
7.61	16.134737	53.163730	3.294986
7.62	16.165631	53.325232	3.298679
7.63	16.196546	53.487043	3.302373
7.64	16.227482	53.649163	3.306068
7.65	16.258439	53.811592	3.309764
7.66	16.289417	53.974332	3.313460
7.67	16.320416	54.137381	3.317157
7.68	16.351436	54.300740	3.320854
7.69	16.382477	54.464410	3.324553
7.70	16.413539	54.628390	3.328252
7.71	16.444621	54.792680	3.331952
7.72	16.475724	54.957282	3.335652
7.73	16.506848	55.122195	3.339353
7.74	16.537993	55.287419	3.343055
7.75	16.569159	55.452955	3.346757
7.76	16.600346	55.618802	3.350460
7.77	16.631553	55.784962	3.354164
7.78	16.662781	55.951434	3.357869
7.79	16.694030	56.118218	3.361574
7.80	16.725300	56.285314	3.365280
7.81	16.756590	56.452724	3.368986
7.82	16.787901	56.620446	3.372694
7.83	16.819232	56.788482	3.376401
7.84	16.850585	56.956831	3.380110
7.85	16.881958	57.125493	3.383819
7.86	16.913351	57.294470	3.387529
7.87	16.944766	57.463761	3.391240
7.88	16.976200	57.633365	3.394951
7.89	17.007656	57.803285	3.398663
7.90	17.039132	57.973519	3.402375
7.91	17.070629	58.144067	3.406088
7.92	17.102146	58.314931	3.409802
7.93	17.133683	58.486110	3.413516
7.94	17.165242	58.657605	3.417232
7.95	17.196820	58.829415	3.420947
7.96	17.228419	59.001541	3.424664
7.97	17.260039	59.173984	3.428381
7.98	17.291679	59.346742	3.432098
7.99	17.323340	59.519817	3.435817
8.00	17.355021	59.693209	3.439535

$\alpha = \mu/kT$	$U_{1/2}(\alpha)$	$U_{3/2}(\alpha)$	$U_{3/2}(\alpha)/U_{1/2}(\alpha)$
8.01	17.386722	59.866918	3.443255
8.02	17.418444	60.040943	3.446975
8.03	17.450186	60.215287	3.450696
8.04	17.481949	60.389947	3.454417
8.05	17.513732	60.564926	3.458139
8.06	17.545536	60.740222	3.461862
8.07	17.577359	60.915836	3.465585
8.08	17.609203	61.091769	3.469309
8.09	17.641068	61.268021	3.473034
8.10	17.672952	61.444501	3.476759
8.11	17.704857	61.621480	3.480484
8.12	17.736782	61.798688	3.484211
8.13	17.768728	61.976215	3.487938
8.14	17.800693	62.154062	3.491665
8.15	17.832679	62.332229	3.495393
8.16	17.864685	62.510716	3.499122
8.17	17.896712	62.689523	3.502851
8.18	17.928758	62.868650	3.506581
8.19	17.960825	63.048098	3.510312
8.20	17.992912	63.227867	3.514043
8.21	18.025019	63.407957	3.517775
8.22	18.057146	63.588367	3.521507
8.23	18.089293	63.769099	3.525240
8.24	18.121460	63.950153	3.528973
8.25	18.153648	64.131529	3.532708
8.26	18.185855	64.313226	3.536442
8.27	18.218083	64.495246	3.540177
8.28	18.250330	64.677588	3.543913
8.29	18.282598	64.860253	3.547650
8.30	18.314886	65.043240	3.551387
8.31	18.347193	65.226550	3.555124
8.32	18.379521	65.410184	3.558862
8.33	18.411869	65.594141	3.562601
8.34	18.444237	65.778421	3.566340
8.35	18.476624	65.963026	3.570080
8.36	18.509032	66.147954	3.573820
8.37	18.541459	66.333206	3.577561
8.38	18.573907	66.518783	3.581303
8.39	18.606374	66.704684	3.585045
8.40	18.638861	66.890911	3.588787
8.41	18.671368	67.077462	3.592531
8.42	18.703895	67.264338	3.596274
8.43	18.736442	67.451540	3.600019
8.44	18.769008	67.639067	3.603763
8.45	18.801595	67.826920	3.607509
8.46	18.834201	68.015099	3.611255
8.47	18.866827	68.203604	3.615001
8.48	18.899473	68.392435	3.618748
8.49	18.932139	68.581593	3.622496
8.50	18.964824	68.771078	3.626244

$\alpha = \mu/kT$	$U_{1/2}(\alpha)$	$U_{3/2}(\alpha)$	$U_{3/2}(\alpha)/U_{1/2}(\alpha)$
8.51	18.997529	68.960890	3.629993
8.52	19.030254	69.151029	3.633742
8.53	19.062999	69.341405	3.637491
8.54	19.095763	69.532289	3.641242
8.55	19.128547	69.723410	3.644992
8.56	19.161351	69.914840	3.648744
8.57	19.194174	70.106638	3.652496
8.58	19.227018	70.298743	3.656248
8.59	19.259880	70.491178	3.660001
8.60	19.292763	70.683941	3.663754
8.61	19.325665	70.877033	3.667508
8.62	19.358586	71.070454	3.671263
8.63	19.391527	71.264205	3.675018
8.64	19.424488	71.458285	3.678773
8.65	19.457469	71.652605	3.682529
8.66	19.490468	71.847434	3.686286
8.67	19.523488	72.042504	3.690043
8.68	19.556527	72.237904	3.693800
8.69	19.589585	72.433635	3.697558
8.70	19.622664	72.629606	3.701317
8.71	19.655761	72.826088	3.705076
8.72	19.688878	73.022811	3.708836
8.73	19.722015	73.219866	3.712596
8.74	19.755171	73.417252	3.716356
8.75	19.788346	73.614969	3.720117
8.76	19.821541	73.813019	3.723879
8.77	19.854755	74.011400	3.727641
8.78	19.887989	74.210114	3.731404
8.79	19.921242	74.409160	3.735167
8.80	19.954515	74.608539	3.738930
8.81	19.987806	74.808260	3.742694
8.82	20.021118	75.008295	3.746459
8.83	20.054448	75.208673	3.750224
8.84	20.087798	75.409384	3.753989
8.85	20.121168	75.610429	3.757756
8.86	20.154556	75.811807	3.761522
8.87	20.187964	76.013520	3.765289
8.88	20.221391	76.215567	3.769056
8.89	20.254838	76.417948	3.772824
8.90	20.288304	76.620663	3.776593
8.91	20.321789	76.823714	3.780362
8.92	20.355293	77.027009	3.784131
8.93	20.388817	77.230820	3.787901
8.94	20.422359	77.434876	3.791671
8.95	20.455921	77.639267	3.795442
8.96	20.489503	77.843904	3.799213
8.97	20.523103	78.049067	3.802985
8.98	20.556722	78.254456	3.806757
8.99	20.590361	78.460102	3.810530
9.00	20.624019	78.666263	3.814303

$\alpha = \mu/kT$	$U_{1/2}(\alpha)$	$U_{3/2}(\alpha)$	$U_{3/2}(\alpha)/U_{1/2}(\alpha)$
9.01	20.657696	78.872672	3.818077
9.02	20.691392	79.079417	3.821851
9.03	20.725108	79.286500	3.825626
9.04	20.758842	79.493920	3.829401
9.05	20.792595	79.701677	3.833176
9.06	20.826368	79.909772	3.836952
9.07	20.860160	80.118204	3.840728
9.08	20.893970	80.326975	3.844505
9.09	20.927800	80.536084	3.848282
9.10	20.961649	80.745531	3.852060
9.11	20.995517	80.955317	3.855838
9.12	21.029404	81.165441	3.859617
9.13	21.063310	81.375905	3.863396
9.14	21.097234	81.586707	3.867175
9.15	21.131178	81.797850	3.870955
9.16	21.165141	82.009331	3.874736
9.17	21.199123	82.221152	3.878517
9.18	21.233124	82.433314	3.882298
9.19	21.267143	82.645815	3.886080
9.20	21.301182	82.858656	3.889862
9.21	21.335240	83.071839	3.893645
9.22	21.369316	83.285361	3.897428
9.23	21.403411	83.499225	3.901211
9.24	21.437526	83.713430	3.904995
9.25	21.471659	83.927976	3.908779
9.26	21.505811	84.142863	3.912564
9.27	21.539982	84.358092	3.916349
9.28	21.574171	84.573663	3.920135
9.29	21.608380	84.789575	3.923921
9.30	21.642607	85.005830	3.927708
9.31	21.676853	85.222427	3.931494
9.32	21.711118	85.439367	3.935282
9.33	21.745402	85.656650	3.939070
9.34	21.779704	85.874275	3.942858
9.35	21.814026	86.092244	3.946646
9.36	21.848366	86.310556	3.950435
9.37	21.882724	86.529211	3.954225
9.38	21.917102	86.748210	3.958015
9.39	21.951498	86.967553	3.961805
9.40	21.985913	87.187240	3.965596
9.41	22.020347	87.407272	3.969387
9.42	22.054799	87.627647	3.973178
9.43	22.089270	87.848368	3.976970
9.44	22.123760	88.069433	3.980763
9.45	22.158268	88.290843	3.984555
9.46	22.192795	88.512598	3.988348
9.47	22.227341	88.734699	3.992142
9.48	22.261905	88.957145	3.995936
9.49	22.296488	89.179937	3.999730
9.50	22.331089	89.403075	4.003525

$\alpha = \mu/kT$	$U_{1/2}(\alpha)$	$U_{3/2}(\alpha)$	$U_{3/2}(\alpha)/U_{1/2}(\alpha)$
9.51	22.365709	89.626569	4.007320
9.52	22.400348	89.850389	4.011116
9.53	22.435005	90.074566	4.014912
9.54	22.469681	90.299049	4.018708
9.55	22.504375	90.523960	4.022505
9.56	22.539088	90.749177	4.026302
9.57	22.573819	90.974741	4.030100
9.58	22.608569	91.200653	4.033898
9.59	22.643338	91.426913	4.037696
9.60	22.678125	91.653520	4.041495
9.61	22.712930	91.880475	4.045294
9.62	22.747754	92.107779	4.049093
9.63	22.782597	92.335430	4.052893
9.64	22.817457	92.563431	4.056693
9.65	22.852337	92.791780	4.060494
9.66	22.887235	93.020477	4.064295
9.67	22.922151	93.249524	4.068097
9.68	22.957085	93.478921	4.071898
9.69	22.992038	93.708666	4.075701
9.70	23.027010	93.938761	4.079503
9.71	23.061999	94.169206	4.083306
9.72	23.097008	94.400001	4.087110
9.73	23.132034	94.631147	4.090913
9.74	23.167079	94.862642	4.094717
9.75	23.202142	95.094488	4.098522
9.76	23.237224	95.326685	4.102327
9.77	23.272324	95.559233	4.106132
9.78	23.307442	95.792132	4.109938
9.79	23.342579	96.025382	4.113744
9.80	23.377733	96.258983	4.117550
9.81	23.412907	96.492936	4.121357
9.82	23.448098	96.727241	4.125164
9.83	23.483308	96.961898	4.128971
9.84	23.518536	97.196908	4.132779
9.85	23.553782	97.432269	4.136587
9.86	23.589046	97.667983	4.140396
9.87	23.624329	97.904050	4.144204
9.88	23.659630	98.140470	4.148014
9.89	23.694949	98.377243	4.151823
9.90	23.730286	98.614369	4.155633
9.91	23.765642	98.851848	4.159444
9.92	23.801015	99.089682	4.163254
9.93	23.836407	99.327869	4.167065
9.94	23.871817	99.566410	4.170877
9.95	23.907245	99.805305	4.174689
9.96	23.942691	100.044555	4.178501
9.97	23.978156	100.284159	4.182313
9.98	24.013638	100.524118	4.186126
9.99	24.049139	100.764432	4.189939
10.00	24.084658	101.005101	4.193753

$\alpha = \mu/kT$	$U_{1/2}(\alpha)$	$U_{3/2}(\alpha)$	$U_{3/2}(\alpha)/U_{1/2}(\alpha)$
10.1	2.444084E+01	1.034314E+02	4.231907E+00
10.2	2.4798[illegible]E+01	1.058933E+02	4.270095E+00
10.3	2.51585[illegible]E+01	1.083312E+02	4.308317E+00
10.4	2.552014E+01	1.1[illegible]9251E+02	4.346571E+00
10.5	2.588347E+01	1.134953E+02	4.384850E+00
10.6	2.624855E+01	1.161019E+02	4.423171E+00
10.7	2.661540E+01	1.187450E+02	4.461517E+00
10.8	2.698398E+01	1.214250E+02	4.499891E+00
10.9	2.735431E+01	1.241419E+02	4.538294E+00
11.0	2.772636E+01	1.268959E+02	4.576724E+00
11.1	2.810013E+01	1.296872E+02	4.615182E+00
11.2	2.847562E+01	1.325160E+02	4.653665E+00
11.3	2.885281E+01	1.353824E+02	4.692174E+00
11.4	2.923170E+01	1.382866E+02	4.730708E+00
11.5	2.961227E+01	1.412288E+02	4.769266E+00
11.6	2.999453E+01	1.442091E+02	4.807848E+00
11.7	3.037845E+01	1.472278E+02	4.846453E+00
11.8	3.076405E+01	1.502849E+02	4.885081E+00
11.9	3.115130E+01	1.533806E+02	4.923731E+00
12.0	3.154020E+01	1.565152E+02	4.962402E+00
12.1	3.193075E+01	1.596887E+02	5.001095E+00
12.2	3.232294E+01	1.629014E+02	5.039808E+00
12.3	3.271675E+01	1.661534E+02	5.078541E+00
12.4	3.311219E+01	1.694448E+02	5.117294E+00
12.5	3.350924E+01	1.727759E+02	5.156066E+00
12.6	3.390790E+01	1.761467E+02	5.194856E+00
12.7	3.430817E+01	1.795575E+02	5.233666E+00
12.8	3.471003E+01	1.830084E+02	5.272493E+00
12.9	3.511348E+01	1.864995E+02	5.311337E+00
13.0	3.551852E+01	1.900311E+02	5.350199E+00
13.1	3.592513E+01	1.936033E+02	5.389077E+00
13.2	3.633331E+01	1.972162E+02	5.427972E+00
13.3	3.674306E+01	2.008700E+02	5.466883E+00
13.4	3.715436E+01	2.045649E+02	5.505810E+00
13.5	3.756722E+01	2.083009E+02	5.544752E+00
13.6	3.798163E+01	2.120784E+02	5.583709E+00
13.7	3.839757E+01	2.158973E+02	5.622681E+00
13.8	3.881505E+01	2.197579E+02	5.661668E+00
13.9	3.923406E+01	2.236604E+02	5.700668E+00
14.0	3.965460E+01	2.276048E+02	5.739682E+00
14.1	4.007665E+01	2.315913E+02	5.778710E+00
14.2	4.050021E+01	2.356202E+02	5.817752E+00
14.3	4.092528E+01	2.396914E+02	5.856806E+00
14.4	4.135186E+01	2.438053E+02	5.895873E+00
14.5	4.177992E+01	2.479619E+02	5.934952E+00
14.6	4.220948E+01	2.521613E+02	5.974044E+00
14.7	4.264053E+01	2.564038E+02	6.013148E+00
14.8	4.307305E+01	2.606895E+02	6.052264E+00
14.9	4.350705E+01	2.650185E+02	6.091391E+00
15.0	4.394252E+01	2.693909E+02	6.130529E+00

$\alpha = \mu/kT$	$U_{1/2}(\alpha)$	$U_{3/2}(\alpha)$	$U_{3/2}(\alpha)/U_{1/2}(\alpha)$
15.1	4.437346E+01	2.738070E+02	6.169679E+00
15.2	4.481786E+01	2.782609E+02	6.208839E+00
15.3	4.525771E+01	2.827796E+02	6.248011E+00
15.4	4.569901E+01	2.873145E+02	6.287192E+00
15.5	4.614176E+01	2.919105E+02	6.326365E+00
15.6	4.658594E+01	2.965469E+02	6.365587E+00
15.7	4.703157E+01	3.012277E+02	6.404799E+00
15.8	4.747862E+01	3.059532E+02	6.444021E+00
15.9	4.792710E+01	3.107235E+02	6.483252E+00
16.0	4.837701E+01	3.155387E+02	6.522493E+00
16.1	4.882833E+01	3.203989E+02	6.561743E+00
16.2	4.928106E+01	3.253044E+02	6.601002E+00
16.3	4.973521E+01	3.302552E+02	6.640270E+00
16.4	5.019075E+01	3.352515E+02	6.679547E+00
16.5	5.064770E+01	3.402934E+02	6.718832E+00
16.6	5.110604E+01	3.453811E+02	6.758126E+00
16.7	5.156578E+01	3.505147E+02	6.797428E+00
16.8	5.202690E+01	3.556943E+02	6.836738E+00
16.9	5.248940E+01	3.609201E+02	6.876056E+00
17.0	5.295329E+01	3.661922E+02	6.915382E+00
17.1	5.341854E+01	3.715108E+02	6.954716E+00
17.2	5.388517E+01	3.768760E+02	6.994057E+00
17.3	5.435316E+01	3.822879E+02	7.033406E+00
17.4	5.482252E+01	3.877466E+02	7.072762E+00
17.5	5.529323E+01	3.932524E+02	7.112126E+00
17.6	5.576530E+01	3.988053E+02	7.151496E+00
17.7	5.623872E+01	4.044055E+02	7.190874E+00
17.8	5.671348E+01	4.100531E+02	7.230258E+00
17.9	5.718959E+01	4.157483E+02	7.269649E+00
18.0	5.766704E+01	4.214911E+02	7.309047E+00
18.1	5.814582E+01	4.272817E+02	7.348451E+00
18.2	5.862593E+01	4.331203E+02	7.387862E+00
18.3	5.910737E+01	4.390069E+02	7.427279E+00
18.4	5.959014E+01	4.449418E+02	7.466702E+00
18.5	6.007422E+01	4.509250E+02	7.506132E+00
18.6	6.055962E+01	4.569567E+02	7.545567E+00
18.7	6.104634E+01	4.630370E+02	7.585009E+00
18.8	6.153436E+01	4.691660E+02	7.624456E+00
18.9	6.202369E+01	4.753439E+02	7.663909E+00
19.0	6.251432E+01	4.815708E+02	7.703367E+00
19.1	6.300625E+01	4.878468E+02	7.742832E+00
19.2	6.349948E+01	4.941721E+02	7.782302E+00
19.3	6.399399E+01	5.005467E+02	7.821777E+00
19.4	6.448980E+01	5.069709E+02	7.861257E+00
19.5	6.498689E+01	5.134447E+02	7.900743E+00
19.6	6.548527E+01	5.199683E+02	7.940234E+00
19.7	6.598492E+01	5.265418E+02	7.979730E+00
19.8	6.648585E+01	5.331654E+02	8.019231E+00
19.9	6.698805E+01	5.398390E+02	8.058737E+00
20.0	6.749152E+01	5.465630E+02	8.098248E+00

$\alpha = \mu/kT$	$U_{1/2}(\alpha)$	$U_{3/2}(\alpha)$	$U_{3/2}(\alpha)/U_{1/2}(\alpha)$
20.1	6.799625E+01	5.533374E+02	8.137763E+00
20.2	6.850225E+01	5.601623E+02	8.177264E+00
20.3	6.900995E+01	5.670379E+02	8.216809E+00
20.4	6.951801E+01	5.739642E+02	8.256338E+00
20.5	7.002778E+01	5.809415E+02	8.295872E+00
20.6	7.053879E+01	5.879698E+02	8.335411E+00
20.7	7.105105E+01	5.950493E+02	8.374954E+00
20.8	7.156456E+01	6.021801E+02	8.414501E+00
20.9	7.207930E+01	6.093623E+02	8.454053E+00
21.0	7.259529E+01	6.165960E+02	8.493609E+00
21.1	7.311251E+01	6.238814E+02	8.533169E+00
21.2	7.363096E+01	6.312185E+02	8.572733E+00
21.3	7.415063E+01	6.386076E+02	8.612301E+00
21.4	7.467154E+01	6.460487E+02	8.651873E+00
21.5	7.519367E+01	6.535419E+02	8.691449E+00
21.6	7.571701E+01	6.610875E+02	8.731029E+00
21.7	7.624158E+01	6.686854E+02	8.770613E+00
21.8	7.676736E+01	6.763358E+02	8.810201E+00
21.9	7.729435E+01	6.840389E+02	8.849792E+00
22.0	7.782255E+01	6.917947E+02	8.889387E+00
22.1	7.835195E+01	6.996035E+02	8.928985E+00
22.2	7.888256E+01	7.074652E+02	8.968588E+00
22.3	7.941437E+01	7.153800E+02	9.008193E+00
22.4	7.994738E+01	7.233481E+02	9.047802E+00
22.5	8.048158E+01	7.313695E+02	9.087415E+00
22.6	8.101697E+01	7.394444E+02	9.127031E+00
22.7	8.155356E+01	7.475730E+02	9.166650E+00
22.8	8.209133E+01	7.557552E+02	9.206273E+00
22.9	8.263029E+01	7.639913E+02	9.245899E+00
23.0	8.317042E+01	7.722813E+02	9.285528E+00
23.1	8.371174E+01	7.806254E+02	9.325160E+00
23.2	8.425423E+01	7.890237E+02	9.364796E+00
23.3	8.479790E+01	7.974763E+02	9.404434E+00
23.4	8.534274E+01	8.059833E+02	9.444076E+00
23.5	8.588875E+01	8.145449E+02	9.483720E+00
23.6	8.643592E+01	8.231611E+02	9.523368E+00
23.7	8.698426E+01	8.318321E+02	9.563018E+00
23.8	8.753376E+01	8.405580E+02	9.602672E+00
23.9	8.808442E+01	8.493389E+02	9.642328E+00
24.0	8.863623E+01	8.581749E+02	9.681987E+00
24.1	8.918920E+01	8.670661E+02	9.721649E+00
24.2	8.974332E+01	8.760128E+02	9.761314E+00
24.3	9.029860E+01	8.850148E+02	9.800981E+00
24.4	9.085501E+01	8.940725E+02	9.840651E+00
24.5	9.141258E+01	9.031859E+02	9.880324E+00
24.6	9.197128E+01	9.123551E+02	9.920000E+00
24.7	9.253112E+01	9.215802E+02	9.959678E+00
24.8	9.309211E+01	9.308613E+02	9.999358E+00
24.9	9.365422E+01	9.401986E+02	1.003904E+01
25.0	9.421747E+01	9.495922E+02	1.007873E+01

$\alpha = \mu/kT$	$U_{1/2}(\alpha)$	$U_{3/2}(\alpha)$	$U_{3/2}(\alpha)/U_{1/2}(\alpha)$
25.1	9.478185E+01	9.590422E+02	1.011842E+01
25.2	9.534736E+01	9.685486E+02	1.015811E+01
25.3	9.591400E+01	9.781117E+02	1.019780E+01
25.4	9.648176E+01	9.877315E+02	1.023749E+01
25.5	9.705064E+01	9.974081E+02	1.027719E+01
25.6	9.762064E+01	1.007142E+03	1.031689E+01
25.7	9.819176E+01	1.016932E+03	1.035659E+01
25.8	9.876399E+01	1.026780E+03	1.039630E+01
25.9	9.933733E+01	1.036685E+03	1.043601E+01
26.0	9.991179E+01	1.046648E+03	1.047572E+01
26.1	1.004874E+02	1.056667E+03	1.051543E+01
26.2	1.010640E+02	1.066745E+03	1.055514E+01
26.3	1.016418E+02	1.076880E+03	1.059486E+01
26.4	1.022207E+02	1.087073E+03	1.063457E+01
26.5	1.028007E+02	1.097324E+03	1.067429E+01
26.6	1.033817E+02	1.107634E+03	1.071402E+01
26.7	1.039639E+02	1.118001E+03	1.075374E+01
26.8	1.045472E+02	1.128426E+03	1.079347E+01
26.9	1.051315E+02	1.138910E+03	1.083320E+01
27.0	1.057170E+02	1.149453E+03	1.087293E+01
27.1	1.063035E+02	1.160054E+03	1.091266E+01
27.2	1.068911E+02	1.170713E+03	1.095239E+01
27.3	1.074798E+02	1.181432E+03	1.099213E+01
27.4	1.080696E+02	1.192209E+03	1.103187E+01
27.5	1.086605E+02	1.203046E+03	1.107161E+01
27.6	1.092524E+02	1.213942E+03	1.111135E+01
27.7	1.098454E+02	1.224896E+03	1.115109E+01
27.8	1.104395E+02	1.235911E+03	1.119084E+01
27.9	1.110347E+02	1.246984E+03	1.123058E+01
28.0	1.116309E+02	1.258118E+03	1.127033E+01
28.1	1.122282E+02	1.269311E+03	1.131008E+01
28.2	1.128266E+02	1.280563E+03	1.134984E+01
28.3	1.134260E+02	1.291876E+03	1.138959E+01
28.4	1.140265E+02	1.303249E+03	1.142935E+01
28.5	1.146281E+02	1.314681E+03	1.146911E+01
28.6	1.152307E+02	1.326174E+03	1.150886E+01
28.7	1.158343E+02	1.337727E+03	1.154863E+01
28.8	1.164391E+02	1.349341E+03	1.158839E+01
28.9	1.170448E+02	1.361015E+03	1.162815E+01
29.0	1.176517E+02	1.372750E+03	1.166792E+01
29.1	1.182595E+02	1.384546E+03	1.170769E+01
29.2	1.188685E+02	1.396402E+03	1.174746E+01
29.3	1.194784E+02	1.408319E+03	1.178723E+01
29.4	1.200895E+02	1.420298E+03	1.182700E+01
29.5	1.207015E+02	1.432337E+03	1.186677E+01
29.6	1.213146E+02	1.444438E+03	1.190655E+01
29.7	1.219287E+02	1.456600E+03	1.194632E+01
29.8	1.225439E+02	1.468824E+03	1.198610E+01
29.9	1.231601E+02	1.481109E+03	1.202588E+01
30.0	1.237774E+02	1.493456E+03	1.206566E+01

$\alpha = \mu/kT$	$U_{1/2}(\alpha)$	$U_{3/2}(\alpha)$	$U_{3/2}(\alpha)/U_{1/2}(\alpha)$
30.1	1.243956E+02	1.505865E+03	1.210545E+01
30.2	1.250149E+02	1.518335E+03	1.214523E+01
30.3	1.256353E+02	1.530866E+03	1.218502E+01
30.4	1.262566E+02	1.543462E+03	1.222480E+01
30.5	1.268790E+02	1.556119E+03	1.226459E+01
30.6	1.275024E+02	1.568838E+03	1.230438E+01
30.7	1.281268E+02	1.581620E+03	1.234417E+01
30.8	1.287523E+02	1.594463E+03	1.238397E+01
30.9	1.293787E+02	1.607370E+03	1.242376E+01
31.0	1.300062E+02	1.620339E+03	1.246355E+01
31.1	1.306347E+02	1.633371E+03	1.250335E+01
31.2	1.312642E+02	1.646466E+03	1.254315E+01
31.3	1.318947E+02	1.659624E+03	1.258295E+01
31.4	1.325262E+02	1.672845E+03	1.262275E+01
31.5	1.331588E+02	1.686129E+03	1.266255E+01
31.6	1.337923E+02	1.699477E+03	1.270235E+01
31.7	1.344269E+02	1.712888E+03	1.274216E+01
31.8	1.350624E+02	1.726362E+03	1.278196E+01
31.9	1.356990E+02	1.739900E+03	1.282177E+01
32.0	1.363365E+02	1.753502E+03	1.286158E+01
32.1	1.369751E+02	1.767168E+03	1.290138E+01
32.2	1.376146E+02	1.780897E+03	1.294119E+01
32.3	1.382551E+02	1.794691E+03	1.298101E+01
32.4	1.388967E+02	1.808548E+03	1.302082E+01
32.5	1.395392E+02	1.822470E+03	1.306063E+01
32.6	1.401827E+02	1.836456E+03	1.310045E+01
32.7	1.408272E+02	1.850507E+03	1.314026E+01
32.8	1.414727E+02	1.864622E+03	1.318008E+01
32.9	1.421192E+02	1.878801E+03	1.321990E+01
33.0	1.427667E+02	1.893046E+03	1.325972E+01
33.1	1.434151E+02	1.907355E+03	1.329954E+01
33.2	1.440646E+02	1.921729E+03	1.333936E+01
33.3	1.447150E+02	1.936168E+03	1.337918E+01
33.4	1.453664E+02	1.950672E+03	1.341900E+01
33.5	1.460187E+02	1.965241E+03	1.345883E+01
33.6	1.466721E+02	1.979875E+03	1.349865E+01
33.7	1.473264E+02	1.994575E+03	1.353848E+01
33.8	1.479817E+02	2.009341E+03	1.357831E+01
33.9	1.486379E+02	2.024172E+03	1.361814E+01
34.0	1.492952E+02	2.039068E+03	1.365796E+01
34.1	1.499534E+02	2.054031E+03	1.369780E+01
34.2	1.506125E+02	2.069059E+03	1.373763E+01
34.3	1.512727E+02	2.084153E+03	1.377746E+01
34.4	1.519338E+02	2.099314E+03	1.381729E+01
34.5	1.525958E+02	2.114540E+03	1.385713E+01
34.6	1.532589E+02	2.129833E+03	1.389696E+01
34.7	1.539229E+02	2.145192E+03	1.393680E+01
34.8	1.545878E+02	2.160617E+03	1.397664E+01
34.9	1.552537E+02	2.176109E+03	1.401647E+01
35.0	1.559205E+02	2.191668E+03	1.405631E+01

$\alpha = \mu/kT$	$U_{1/2}(\alpha)$	$U_{3/2}(\alpha)$	$U_{3/2}(\alpha)/U_{1/2}(\alpha)$
35.1	1.565884E+02	2.207294E+03	1.409615E+01
35.2	1.572571E+02	2.222986E+03	1.413599E+01
35.3	1.579268E+02	2.238745E+03	1.417584E+01
35.4	1.585975E+02	2.254571E+03	1.421568E+01
35.5	1.592691E+02	2.270465E+03	1.425552E+01
35.6	1.599417E+02	2.286425E+03	1.429537E+01
35.7	1.606152E+02	2.302453E+03	1.433521E+01
35.8	1.612896E+02	2.318548E+03	1.437506E+01
35.9	1.619650E+02	2.334711E+03	1.441491E+01
36.0	1.626414E+02	2.350941E+03	1.445475E+01
36.1	1.633187E+02	2.367239E+03	1.449460E+01
36.2	1.639969E+02	2.383605E+03	1.453445E+01
36.3	1.646761E+02	2.400039E+03	1.457430E+01
36.4	1.653562E+02	2.416540E+03	1.461415E+01
36.5	1.660372E+02	2.433110E+03	1.465401E+01
36.6	1.667192E+02	2.449748E+03	1.469386E+01
36.7	1.674021E+02	2.466454E+03	1.473371E+01
36.8	1.680859E+02	2.483228E+03	1.477357E+01
36.9	1.687707E+02	2.500071E+03	1.481342E+01
37.0	1.694564E+02	2.516982E+03	1.485328E+01
37.1	1.701430E+02	2.533962E+03	1.489313E+01
37.2	1.708305E+02	2.551011E+03	1.493299E+01
37.3	1.715190E+02	2.568128E+03	1.497285E+01
37.4	1.722084E+02	2.585315E+03	1.501271E+01
37.5	1.728987E+02	2.602570E+03	1.505257E+01
37.6	1.735900E+02	2.619895E+03	1.509243E+01
37.7	1.742821E+02	2.637288E+03	1.513229E+01
37.8	1.749752E+02	2.654751E+03	1.517215E+01
37.9	1.756692E+02	2.672283E+03	1.521202E+01
38.0	1.763642E+02	2.689885E+03	1.525188E+01
38.1	1.770600E+02	2.707556E+03	1.529174E+01
38.2	1.777567E+02	2.725297E+03	1.533161E+01
38.3	1.784544E+02	2.743107E+03	1.537147E+01
38.4	1.791530E+02	2.760988E+03	1.541134E+01
38.5	1.798525E+02	2.778938E+03	1.545121E+01
38.6	1.805529E+02	2.796958E+03	1.549107E+01
38.7	1.812542E+02	2.815049E+03	1.553094E+01
38.8	1.819564E+02	2.833209E+03	1.557081E+01
38.9	1.826595E+02	2.851440E+03	1.561068E+01
39.0	1.833636E+02	2.869741E+03	1.565055E+01
39.1	1.840685E+02	2.888113E+03	1.569042E+01
39.2	1.847743E+02	2.906555E+03	1.573029E+01
39.3	1.854811E+02	2.925068E+03	1.577017E+01
39.4	1.861887E+02	2.943651E+03	1.581004E+01
39.5	1.868973E+02	2.962305E+03	1.584991E+01
39.6	1.876067E+02	2.981031E+03	1.588979E+01
39.7	1.883170E+02	2.999827E+03	1.592966E+01
39.8	1.890283E+02	3.018694E+03	1.596954E+01
39.9	1.897404E+02	3.037632E+03	1.600941E+01
40.0	1.904534E+02	3.056642E+03	1.604929E+01

$\alpha = \mu/kT$	$U_{1/2}(\alpha)$	$U_{3/2}(\alpha)$	$U_{3/2}(\alpha)/U_{1/2}(\alpha)$
40.1	1.911673E+02	3.075723E+03	1.608917E+01
40.2	1.918821E+02	3.094876E+03	1.612905E+01
40.3	1.925978E+02	3.114100E+03	1.616893E+01
40.4	1.933144E+02	3.133395E+03	1.620880E+01
40.5	1.940319E+02	3.152762E+03	1.624868E+01
40.6	1.947502E+02	3.172202E+03	1.628856E+01
40.7	1.954695E+02	3.191713E+03	1.632845E+01
40.8	1.961896E+02	3.211296E+03	1.636833E+01
40.9	1.969106E+02	3.230951E+03	1.640821E+01
41.0	1.976325E+02	3.250678E+03	1.644809E+01
41.1	1.983553E+02	3.270477E+03	1.648797E+01
41.2	1.990789E+02	3.290349E+03	1.652786E+01
41.3	1.998035E+02	3.310293E+03	1.656774E+01
41.4	2.005289E+02	3.330309E+03	1.660763E+01
41.5	2.012552E+02	3.350399E+03	1.664751E+01
41.6	2.019824E+02	3.370561E+03	1.668740E+01
41.7	2.027104E+02	3.390795E+03	1.672729E+01
41.8	2.034393E+02	3.411103E+03	1.676717E+01
41.9	2.041691E+02	3.431483E+03	1.680706E+01
42.0	2.048998E+02	3.451936E+03	1.684695E+01
42.1	2.056313E+02	3.472463E+03	1.688684E+01
42.2	2.063637E+02	3.493063E+03	1.692673E+01
42.3	2.070970E+02	3.513736E+03	1.696662E+01
42.4	2.078312E+02	3.534482E+03	1.700651E+01
42.5	2.085662E+02	3.555302E+03	1.704640E+01
42.6	2.093021E+02	3.576195E+03	1.708629E+01
42.7	2.100388E+02	3.597163E+03	1.712618E+01
42.8	2.107764E+02	3.618203E+03	1.716607E+01
42.9	2.115149E+02	3.639318E+03	1.720597E+01
43.0	2.122542E+02	3.660506E+03	1.724586E+01
43.1	2.129944E+02	3.681769E+03	1.728575E+01
43.2	2.137355E+02	3.703105E+03	1.732565E+01
43.3	2.144774E+02	3.724516E+03	1.736554E+01
43.4	2.152201E+02	3.746001E+03	1.740544E+01
43.5	2.159638E+02	3.767560E+03	1.744533E+01
43.6	2.167083E+02	3.789193E+03	1.748523E+01
43.7	2.174536E+02	3.810902E+03	1.752513E+01
43.8	2.181998E+02	3.832684E+03	1.756502E+01
43.9	2.189468E+02	3.854542E+03	1.760492E+01
44.0	2.196947E+02	3.876474E+03	1.764482E+01
44.1	2.204435E+02	3.898480E+03	1.768472E+01
44.2	2.211931E+02	3.920562E+03	1.772462E+01
44.3	2.219435E+02	3.942719E+03	1.776452E+01
44.4	2.226948E+02	3.964951E+03	1.780442E+01
44.5	2.234470E+02	3.987258E+03	1.784432E+01
44.6	2.241999E+02	4.009640E+03	1.788422E+01
44.7	2.249538E+02	4.032098E+03	1.792412E+01
44.8	2.257085E+02	4.054631E+03	1.796402E+01
44.9	2.264640E+02	4.077240E+03	1.800392E+01
45.0	2.272203E+02	4.099924E+03	1.804382E+01

$\alpha = \mu/kT$	$U_{1/2}(\alpha)$	$U_{3/2}(\alpha)$	$U_{3/2}(\alpha)/U_{1/2}(\alpha)$
45.1	2.279776E+02	4.122684E+03	1.808373E+01
45.2	2.287356E+02	4.145520E+03	1.812363E+01
45.3	2.294945E+02	4.168431E+03	1.816354E+01
45.4	2.302542E+02	4.191419E+03	1.820344E+01
45.5	2.310148E+02	4.214482E+03	1.824334E+01
45.6	2.317762E+02	4.237622E+03	1.828325E+01
45.7	2.325384E+02	4.260637E+03	1.832315E+01
45.8	2.333015E+02	4.284129E+03	1.836306E+01
45.9	2.340654E+02	4.307498E+03	1.840297E+01
46.0	2.348301E+02	4.330942E+03	1.844287E+01
46.1	2.355957E+02	4.354464E+03	1.848278E+01
46.2	2.363621E+02	4.378061E+03	1.852269E+01
46.3	2.371293E+02	4.401736E+03	1.856260E+01
46.4	2.378974E+02	4.425487E+03	1.860250E+01
46.5	2.386663E+02	4.449316E+03	1.864241E+01
46.6	2.394360E+02	4.473221E+03	1.868232E+01
46.7	2.402066E+02	4.497203E+03	1.872223E+01
46.8	2.409779E+02	4.521262E+03	1.876214E+01
46.9	2.417501E+02	4.545398E+03	1.880205E+01
47.0	2.425231E+02	4.569612E+03	1.884196E+01
47.1	2.432970E+02	4.593903E+03	1.888187E+01
47.2	2.440717E+02	4.618271E+03	1.892179E+01
47.3	2.448471E+02	4.642717E+03	1.896170E+01
47.4	2.456235E+02	4.667241E+03	1.900161E+01
47.5	2.464006E+02	4.691842E+03	1.904152E+01
47.6	2.471785E+02	4.716521E+03	1.908143E+01
47.7	2.479573E+02	4.741278E+03	1.912135E+01
47.8	2.487369E+02	4.766113E+03	1.916126E+01
47.9	2.495173E+02	4.791025E+03	1.920118E+01
48.0	2.502985E+02	4.816016E+03	1.924109E+01
48.1	2.510805E+02	4.841085E+03	1.928100E+01
48.2	2.518634E+02	4.866232E+03	1.932092E+01
48.3	2.526470E+02	4.891458E+03	1.936084E+01
48.4	2.534315E+02	4.916762E+03	1.940075E+01
48.5	2.542168E+02	4.942144E+03	1.944067E+01
48.6	2.550029E+02	4.967605E+03	1.948058E+01
48.7	2.557898E+02	4.993145E+03	1.952050E+01
48.8	2.565775E+02	5.018763E+03	1.956042E+01
48.9	2.573660E+02	5.044460E+03	1.960033E+01
49.0	2.581553E+02	5.070236E+03	1.964025E+01
49.1	2.589455E+02	5.096091E+03	1.968017E+01
49.2	2.597364E+02	5.122025E+03	1.972009E+01
49.3	2.605282E+02	5.148039E+03	1.976001E+01
49.4	2.613207E+02	5.174131E+03	1.979993E+01
49.5	2.621141E+02	5.200303E+03	1.983985E+01
49.6	2.629082E+02	5.226554E+03	1.987977E+01
49.7	2.637032E+02	5.252884E+03	1.991969E+01
49.8	2.644989E+02	5.279294E+03	1.995961E+01
49.9	2.652955E+02	5.305784E+03	1.999953E+01
50.0	2.660928E+02	5.332354E+03	2.003945E+01

$\alpha = \mu/kT$	$U_{1/2}(\alpha)$	$U_{3/2}(\alpha)$	$U_{3/2}(\alpha)/U_{1/2}(\alpha)$
50.1	2.668910E+02	5.359003E+03	2.007937E+01
50.2	2.676899E+02	5.385732E+03	2.011929E+01
50.3	2.684897E+02	5.412541E+03	2.015921E+01
50.4	2.692902E+02	5.439430E+03	2.019914E+01
50.5	2.700916E+02	5.466399E+03	2.023906E+01
50.6	2.708937E+02	5.493448E+03	2.027898E+01
50.7	2.716966E+02	5.520578E+03	2.031890E+01
50.8	2.725003E+02	5.547787E+03	2.035883E+01
50.9	2.733048E+02	5.575078E+03	2.039875E+01
51.0	2.741101E+02	5.602448E+03	2.043868E+01
51.1	2.749162E+02	5.629900E+03	2.047860E+01
51.2	2.757231E+02	5.657432E+03	2.051853E+01
51.3	2.765308E+02	5.685044E+03	2.055845E+01
51.4	2.773392E+02	5.712738E+03	2.059838E+01
51.5	2.781485E+02	5.740512E+03	2.063830E+01
51.6	2.789585E+02	5.768368E+03	2.067823E+01
51.7	2.797693E+02	5.796304E+03	2.071815E+01
51.8	2.805809E+02	5.824321E+03	2.075808E+01
51.9	2.813933E+02	5.852420E+03	2.079801E+01
52.0	2.822065E+02	5.880600E+03	2.083793E+01
52.1	2.830204E+02	5.908861E+03	2.087786E+01
52.2	2.838352E+02	5.937204E+03	2.091779E+01
52.3	2.846507E+02	5.965629E+03	2.095772E+01
52.4	2.854670E+02	5.994134E+03	2.099764E+01
52.5	2.862841E+02	6.022722E+03	2.103757E+01
52.6	2.871019E+02	6.051391E+03	2.107750E+01
52.7	2.879206E+02	6.080142E+03	2.111743E+01
52.8	2.887400E+02	6.108975E+03	2.115736E+01
52.9	2.895602E+02	6.137890E+03	2.119729E+01
53.0	2.903811E+02	6.166887E+03	2.123722E+01
53.1	2.912029E+02	6.195967E+03	2.127715E+01
53.2	2.920254E+02	6.225128E+03	2.131708E+01
53.3	2.928487E+02	6.254372E+03	2.135701E+01
53.4	2.936727E+02	6.283698E+03	2.139694E+01
53.5	2.944976E+02	6.313106E+03	2.143687E+01
53.6	2.953232E+02	6.342597E+03	2.147680E+01
53.7	2.961495E+02	6.372171E+03	2.151673E+01
53.8	2.969767E+02	6.401827E+03	2.155667E+01
53.9	2.978046E+02	6.431566E+03	2.159660E+01
54.0	2.986333E+02	6.461388E+03	2.163653E+01
54.1	2.994627E+02	6.491293E+03	2.167646E+01
54.2	3.002930E+02	6.521281E+03	2.171640E+01
54.3	3.011239E+02	6.551352E+03	2.175633E+01
54.4	3.019557E+02	6.581506E+03	2.179626E+01
54.5	3.027882E+02	6.611743E+03	2.183620E+01
54.6	3.036215E+02	6.642063E+03	2.187613E+01
54.7	3.044555E+02	6.672467E+03	2.191606E+01
54.8	3.052904E+02	6.702954E+03	2.195600E+01
54.9	3.061259E+02	6.733525E+03	2.199593E+01
55.0	3.069623E+02	6.764180E+03	2.203587E+01

$\alpha = \mu/kT$	$U_{1/2}(\alpha)$	$U_{3/2}(\alpha)$	$U_{3/2}(\alpha)/U_{1/2}(\alpha)$
55.1	3.077994E+02	6.794918E+03	2.207580E+01
55.2	3.086372E+02	6.825739E+03	2.211574E+01
55.3	3.094758E+02	6.856645E+03	2.215567E+01
55.4	3.103152E+02	6.887635E+03	2.219561E+01
55.5	3.111553E+02	6.918708E+03	2.223554E+01
55.6	3.119962E+02	6.949866E+03	2.227548E+01
55.7	3.128379E+02	6.981107E+03	2.231542E+01
55.8	3.136803E+02	7.012433E+03	2.235535E+01
55.9	3.145234E+02	7.043844E+03	2.239529E+01
56.0	3.153673E+02	7.075338E+03	2.243523E+01
56.1	3.162120E+02	7.106917E+03	2.247516E+01
56.2	3.170574E+02	7.138581E+03	2.251510E+01
56.3	3.179036E+02	7.170329E+03	2.255504E+01
56.4	3.187505E+02	7.202161E+03	2.259498E+01
56.5	3.195982E+02	7.234079E+03	2.263492E+01
56.6	3.204466E+02	7.266081E+03	2.267485E+01
56.7	3.212958E+02	7.298168E+03	2.271479E+01
56.8	3.221458E+02	7.330340E+03	2.275473E+01
56.9	3.229964E+02	7.362597E+03	2.279467E+01
57.0	3.238479E+02	7.394939E+03	2.283461E+01
57.1	3.247000E+02	7.427367E+03	2.287455E+01
57.2	3.255530E+02	7.459879E+03	2.291449E+01
57.3	3.264066E+02	7.492477E+03	2.295443E+01
57.4	3.272610E+02	7.525161E+03	2.299437E+01
57.5	3.281162E+02	7.557930E+03	2.303431E+01
57.6	3.289721E+02	7.590784E+03	2.307425E+01
57.7	3.298287E+02	7.623724E+03	2.311419E+01
57.8	3.306861E+02	7.656750E+03	2.315413E+01
57.9	3.315443E+02	7.689861E+03	2.319407E+01
58.0	3.324031E+02	7.723059E+03	2.323401E+01
58.1	3.332627E+02	7.756342E+03	2.327396E+01
58.2	3.341231E+02	7.789711E+03	2.331390E+01
58.3	3.349842E+02	7.823167E+03	2.335384E+01
58.4	3.358460E+02	7.856708E+03	2.339378E+01
58.5	3.367086E+02	7.890336E+03	2.343372E+01
58.6	3.375719E+02	7.924050E+03	2.347367E+01
58.7	3.384359E+02	7.957850E+03	2.351361E+01
58.8	3.393007E+02	7.991737E+03	2.355355E+01
58.9	3.401662E+02	8.025710E+03	2.359349E+01
59.0	3.410325E+02	8.059770E+03	2.363344E+01
59.1	3.418995E+02	8.093917E+03	2.367338E+01
59.2	3.427672E+02	8.128150E+03	2.371332E+01
59.3	3.436357E+02	8.162470E+03	2.375327E+01
59.4	3.445049E+02	8.196877E+03	2.379321E+01
59.5	3.453748E+02	8.231371E+03	2.383316E+01
59.6	3.462454E+02	8.265952E+03	2.387310E+01
59.7	3.471168E+02	8.300620E+03	2.391305E+01
59.8	3.479889E+02	8.335376E+03	2.395299E+01
59.9	3.486618E+02	8.370218E+03	2.399294E+01
60.0	3.497354E+02	8.405148E+03	2.403288E+01

$\alpha = \mu/kT$	$U_{1/2}(\alpha)$	$U_{3/2}(\alpha)$	$U_{3/2}(\alpha)/U_{1/2}(\alpha)$
60.1	3.506097E+02	8.440165E+03	2.407283E+01
60.2	3.514847E+02	8.475270E+03	2.411277E+01
60.3	3.523604E+02	8.510462E+03	2.415272E+01
60.4	3.532369E+02	8.545742E+03	2.419266E+01
60.5	3.541141E+02	8.581110E+03	2.423261E+01
60.6	3.549921E+02	8.616565E+03	2.427256E+01
60.7	3.558707E+02	8.652108E+03	2.431250E+01
60.8	3.567501E+02	8.687739E+03	2.435245E+01
60.9	3.576302E+02	8.723458E+03	2.439240E+01
61.0	3.585111E+02	8.759265E+03	2.443234E+01
61.1	3.593926E+02	8.795160E+03	2.447229E+01
61.2	3.602749E+02	8.831144E+03	2.451224E+01
61.3	3.611579E+02	8.867215E+03	2.455218E+01
61.4	3.620416E+02	8.903375E+03	2.459213E+01
61.5	3.629261E+02	8.939624E+03	2.463208E+01
61.6	3.638112E+02	8.975961E+03	2.467203E+01
61.7	3.646971E+02	9.012386E+03	2.471198E+01
61.8	3.655837E+02	9.048900E+03	2.475192E+01
61.9	3.664710E+02	9.085503E+03	2.479187E+01
62.0	3.673591E+02	9.122194E+03	2.483182E+01
62.1	3.682478E+02	9.158975E+03	2.487177E+01
62.2	3.691373E+02	9.195844E+03	2.491172E+01
62.3	3.700275E+02	9.232802E+03	2.495167E+01
62.4	3.709183E+02	9.269849E+03	2.499162E+01
62.5	3.718100E+02	9.306986E+03	2.503157E+01
62.6	3.727023E+02	9.344211E+03	2.507152E+01
62.7	3.735953E+02	9.381526E+03	2.511147E+01
62.8	3.744891E+02	9.418931E+03	2.515142E+01
62.9	3.753835E+02	9.456424E+03	2.519137E+01
63.0	3.762787E+02	9.494007E+03	2.523132E+01
63.1	3.771746E+02	9.531680E+03	2.527127E+01
63.2	3.780712E+02	9.569442E+03	2.531122E+01
63.3	3.789685E+02	9.607294E+03	2.535117E+01
63.4	3.798665E+02	9.645236E+03	2.539112E+01
63.5	3.807652E+02	9.683267E+03	2.543107E+01
63.6	3.816647E+02	9.721389E+03	2.547102E+01
63.7	3.825648E+02	9.759600E+03	2.551097E+01
63.8	3.834657E+02	9.797902E+03	2.555092E+01
63.9	3.843672E+02	9.836294E+03	2.559088E+01
64.0	3.852695E+02	9.874775E+03	2.563083E+01
64.1	3.861724E+02	9.913348E+03	2.567078E+01
64.2	3.870761E+02	9.952010E+03	2.571073E+01
64.3	3.879805E+02	9.990763E+03	2.575068E+01
64.4	3.888856E+02	1.002961E+04	2.579064E+01
64.5	3.897913E+02	1.006854E+04	2.583059E+01
64.6	3.906978E+02	1.010756E+04	2.587054E+01
64.7	3.916050E+02	1.014668E+04	2.591049E+01
64.8	3.925129E+02	1.018589E+04	2.595045E+01
64.9	3.934215E+02	1.022518E+04	2.599040E+01
65.0	3.943308E+02	1.026457E+04	2.603035E+01

$\alpha = \mu/kT$	$U_{1/2}(\alpha)$	$U_{3/2}(\alpha)$	$U_{3/2}(\alpha)/U_{1/2}(\alpha)$
65.1	3.952408E+02	1.030405E+04	2.607031E+01
65.2	3.961515E+02	1.034362E+04	2.611026E+01
65.3	3.970628E+02	1.038328E+04	2.615021E+01
65.4	3.979749E+02	1.042303E+04	2.619017E+01
65.5	3.988877E+02	1.046287E+04	2.623012E+01
65.6	3.998012E+02	1.050281E+04	2.627008E+01
65.7	4.007154E+02	1.054283E+04	2.631003E+01
65.8	4.016303E+02	1.058295E+04	2.634998E+01
65.9	4.025458E+02	1.062316E+04	2.638994E+01
66.0	4.034621E+02	1.066346E+04	2.642989E+01
66.1	4.043790E+02	1.070385E+04	2.646985E+01
66.2	4.052967E+02	1.074434E+04	2.650980E+01
66.3	4.062151E+02	1.078491E+04	2.654976E+01
66.4	4.071341E+02	1.082558E+04	2.658971E+01
66.5	4.080538E+02	1.086634E+04	2.662967E+01
66.6	4.089743E+02	1.090719E+04	2.666962E+01
66.7	4.098954E+02	1.094813E+04	2.670958E+01
66.8	4.108172E+02	1.098917E+04	2.674954E+01
66.9	4.117397E+02	1.103030E+04	2.678949E+01
67.0	4.126629E+02	1.107152E+04	2.682945E+01
67.1	4.135867E+02	1.111283E+04	2.686940E+01
67.2	4.145113E+02	1.115423E+04	2.690936E+01
67.3	4.154366E+02	1.119573E+04	2.694932E+01
67.4	4.163625E+02	1.123732E+04	2.698927E+01
67.5	4.172891E+02	1.127900E+04	2.702923E+01
67.6	4.182165E+02	1.132078E+04	2.706919E+01
67.7	4.191445E+02	1.136265E+04	2.710914E+01
67.8	4.200732E+02	1.140461E+04	2.714910E+01
67.9	4.210025E+02	1.144666E+04	2.718906E+01
68.0	4.219326E+02	1.148881E+04	2.722902E+01
68.1	4.228633E+02	1.153105E+04	2.726897E+01
68.2	4.237948E+02	1.157338E+04	2.730893E+01
68.3	4.247269E+02	1.161581E+04	2.734889E+01
68.4	4.256597E+02	1.165833E+04	2.738885E+01
68.5	4.265931E+02	1.170094E+04	2.742880E+01
68.6	4.275273E+02	1.174365E+04	2.746876E+01
68.7	4.284621E+02	1.178645E+04	2.750872E+01
68.8	4.293977E+02	1.182934E+04	2.754868E+01
68.9	4.303339E+02	1.187232E+04	2.758864E+01
69.0	4.312707E+02	1.191540E+04	2.762860E+01
69.1	4.322083E+02	1.195858E+04	2.766855E+01
69.2	4.331465E+02	1.200185E+04	2.770851E+01
69.3	4.340855E+02	1.204521E+04	2.774847E+01
69.4	4.350251E+02	1.208866E+04	2.778843E+01
69.5	4.359653E+02	1.213221E+04	2.782839E+01
69.6	4.369063E+02	1.217586E+04	2.786835E+01
69.7	4.378479E+02	1.221959E+04	2.790831E+01
69.8	4.387902E+02	1.226343E+04	2.794827E+01
69.9	4.397332E+02	1.230735E+04	2.798823E+01
70.0	4.406768E+02	1.235137E+04	2.802819E+01

$\alpha = \mu/kT$	$U_{1/2}(\alpha)$	$U_{3/2}(\alpha)$	$U_{3/2}(\alpha)/U_{1/2}(\alpha)$
70.1	4.416217E+02	1.239549E+04	2.806815E+01
70.2	4.425662E+02	1.243970E+04	2.810811E+01
70.3	4.435119E+02	1.248400E+04	2.814807E+01
70.4	4.444582E+02	1.252840E+04	2.818803E+01
70.5	4.454052E+02	1.257289E+04	2.822799E+01
70.6	4.463529E+02	1.261748E+04	2.826795E+01
70.7	4.473013E+02	1.266216E+04	2.830791E+01
70.8	4.482503E+02	1.270694E+04	2.834787E+01
70.9	4.492000E+02	1.275181E+04	2.838783E+01
71.0	4.501504E+02	1.279678E+04	2.842779E+01
71.1	4.511014E+02	1.284184E+04	2.846775E+01
71.2	4.520532E+02	1.288700E+04	2.850771E+01
71.3	4.530055E+02	1.293225E+04	2.854767E+01
71.4	4.539586E+02	1.297760E+04	2.858763E+01
71.5	4.549123E+02	1.302305E+04	2.862760E+01
71.6	4.558667E+02	1.306858E+04	2.866756E+01
71.7	4.568218E+02	1.311422E+04	2.870752E+01
71.8	4.577775E+02	1.315995E+04	2.874748E+01
71.9	4.587339E+02	1.320577E+04	2.878744E+01
72.0	4.596909E+02	1.325170E+04	2.882740E+01
72.1	4.606486E+02	1.329771E+04	2.886737E+01
72.2	4.616070E+02	1.334383E+04	2.890733E+01
72.3	4.625661E+02	1.339003E+04	2.894729E+01
72.4	4.635258E+02	1.343634E+04	2.898725E+01
72.5	4.644862E+02	1.348274E+04	2.902722E+01
72.6	4.654472E+02	1.352924E+04	2.906718E+01
72.7	4.664089E+02	1.357583E+04	2.910714E+01
72.8	4.673712E+02	1.362252E+04	2.914710E+01
72.9	4.683343E+02	1.366930E+04	2.918707E+01
73.0	4.692980E+02	1.371618E+04	2.922703E+01
73.1	4.702623E+02	1.376316E+04	2.926699E+01
73.2	4.712273E+02	1.381024E+04	2.930696E+01
73.3	4.721930E+02	1.385741E+04	2.934692E+01
73.4	4.731593E+02	1.390468E+04	2.938688E+01
73.5	4.741263E+02	1.395204E+04	2.942685E+01
73.6	4.750939E+02	1.399950E+04	2.946681E+01
73.7	4.760622E+02	1.404706E+04	2.950677E+01
73.8	4.770312E+02	1.409471E+04	2.954674E+01
73.9	4.780008E+02	1.414247E+04	2.958670E+01
74.0	4.789710E+02	1.419031E+04	2.962666E+01
74.1	4.799420E+02	1.423826E+04	2.966663E+01
74.2	4.809135E+02	1.428630E+04	2.970659E+01
74.3	4.818858E+02	1.433444E+04	2.974656E+01
74.4	4.828587E+02	1.438268E+04	2.978652E+01
74.5	4.838322E+02	1.443101E+04	2.982649E+01
74.6	4.848064E+02	1.447945E+04	2.986645E+01
74.7	4.857812E+02	1.452798E+04	2.990641E+01
74.8	4.867568E+02	1.457660E+04	2.994638E+01
74.9	4.877329E+02	1.462533E+04	2.998634E+01
75.0	4.887097E+02	1.467415E+04	3.002631E+01

$\alpha = \mu/kT$	$U_{1/2}(\alpha)$	$U_{3/2}(\alpha)$	$U_{3/2}(\alpha)/U_{1/2}(\alpha)$
75.1	4.896872E+02	1.472307E+04	3.006627E+01
75.2	4.906659E+02	1.477209E+04	3.010624E+01
75.3	4.916440E+02	1.482120E+04	3.014620E+01
75.4	4.926235E+02	1.487042E+04	3.018617E+01
75.5	4.936035E+02	1.491973E+04	3.022613E+01
75.6	4.945842E+02	1.496914E+04	3.026610E+01
75.7	4.955656E+02	1.501864E+04	3.030607E+01
75.8	4.965476E+02	1.506825E+04	3.034603E+01
75.9	4.975303E+02	1.511795E+04	3.038600E+01
76.0	4.985136E+02	1.516776E+04	3.042596E+01
76.1	4.994975E+02	1.521766E+04	3.046593E+01
76.2	5.004821E+02	1.526765E+04	3.050589E+01
76.3	5.014674E+02	1.531775E+04	3.054586E+01
76.4	5.024532E+02	1.536795E+04	3.058583E+01
76.5	5.034398E+02	1.541824E+04	3.062579E+01
76.6	5.044270E+02	1.546864E+04	3.066576E+01
76.7	5.054148E+02	1.551913E+04	3.070573E+01
76.8	5.064033E+02	1.556972E+04	3.074569E+01
76.9	5.073924E+02	1.562041E+04	3.078566E+01
77.0	5.083821E+02	1.567120E+04	3.082563E+01
77.1	5.093725E+02	1.572209E+04	3.086559E+01
77.2	5.103636E+02	1.577307E+04	3.090556E+01
77.3	5.113553E+02	1.582416E+04	3.094553E+01
77.4	5.123476E+02	1.587534E+04	3.098549E+01
77.5	5.133406E+02	1.592663E+04	3.102546E+01
77.6	5.143342E+02	1.597801E+04	3.106543E+01
77.7	5.153284E+02	1.602949E+04	3.110540E+01
77.8	5.163233E+02	1.608108E+04	3.114536E+01
77.9	5.173189E+02	1.613276E+04	3.118533E+01
78.0	5.183150E+02	1.618454E+04	3.122530E+01
78.1	5.193118E+02	1.623642E+04	3.126527E+01
78.2	5.203093E+02	1.628840E+04	3.130523E+01
78.3	5.213074E+02	1.634048E+04	3.134520E+01
78.4	5.223061E+02	1.639266E+04	3.138517E+01
78.5	5.233055E+02	1.644495E+04	3.142514E+01
78.6	5.243054E+02	1.649733E+04	3.146510E+01
78.7	5.253061E+02	1.654981E+04	3.150507E+01
78.8	5.263074E+02	1.660239E+04	3.154504E+01
78.9	5.273093E+02	1.665507E+04	3.158501E+01
79.0	5.283118E+02	1.670785E+04	3.162498E+01
79.1	5.293150E+02	1.676073E+04	3.166495E+01
79.2	5.303188E+02	1.681371E+04	3.170491E+01
79.3	5.313232E+02	1.686679E+04	3.174488E+01
79.4	5.323283E+02	1.691998E+04	3.178485E+01
79.5	5.333340E+02	1.697326E+04	3.182482E+01
79.6	5.343404E+02	1.702664E+04	3.186479E+01
79.7	5.353473E+02	1.708013E+04	3.190476E+01
79.8	5.363550E+02	1.713371E+04	3.194473E+01
79.9	5.373632E+02	1.718740E+04	3.198470E+01
80.0	5.383721E+02	1.724119E+04	3.202467E+01

$\alpha = \mu/kT$	$U_{1/2}(\alpha)$	$U_{3/2}(\alpha)$	$U_{3/2}(\alpha)/U_{1/2}(\alpha)$
80.1	5.393816E+02	1.729507E+04	3.206463E+01
80.2	5.403917E+02	1.734906E+04	3.210460E+01
80.3	5.414025E+02	1.740315E+04	3.214457E+01
80.4	5.424139E+02	1.745734E+04	3.218454E+01
80.5	5.434259E+02	1.751163E+04	3.222451E+01
80.6	5.444385E+02	1.756603E+04	3.226448E+01
80.7	5.454518E+02	1.762052E+04	3.230445E+01
80.8	5.464657E+02	1.767512E+04	3.234442E+01
80.9	5.474803E+02	1.772982E+04	3.238439E+01
81.0	5.484954E+02	1.778461E+04	3.242436E+01
81.1	5.495112E+02	1.783951E+04	3.246433E+01
81.2	5.505276E+02	1.789452E+04	3.250430E+01
81.3	5.515447E+02	1.794962E+04	3.254427E+01
81.4	5.525624E+02	1.800483E+04	3.258424E+01
81.5	5.535807E+02	1.806013E+04	3.262421E+01
81.6	5.545996E+02	1.811554E+04	3.266418E+01
81.7	5.556191E+02	1.817105E+04	3.270415E+01
81.8	5.566393E+02	1.822667E+04	3.274412E+01
81.9	5.576601E+02	1.828238E+04	3.278409E+01
82.0	5.586815E+02	1.833820E+04	3.282406E+01
82.1	5.597035E+02	1.839412E+04	3.286403E+01
82.2	5.607262E+02	1.845014E+04	3.290401E+01
82.3	5.617495E+02	1.850626E+04	3.294398E+01
82.4	5.627734E+02	1.856249E+04	3.298395E+01
82.5	5.637979E+02	1.861882E+04	3.302392E+01
82.6	5.648231E+02	1.867525E+04	3.306389E+01
82.7	5.658488E+02	1.873178E+04	3.310386E+01
82.8	5.668752E+02	1.878842E+04	3.314383E+01
82.9	5.679022E+02	1.884516E+04	3.318380E+01
83.0	5.689299E+02	1.890200E+04	3.322377E+01
83.1	5.699581E+02	1.895894E+04	3.326375E+01
83.2	5.709870E+02	1.901599E+04	3.330372E+01
83.3	5.720165E+02	1.907314E+04	3.334369E+01
83.4	5.730466E+02	1.913039E+04	3.338366E+01
83.5	5.740773E+02	1.918775E+04	3.342363E+01
83.6	5.751087E+02	1.924521E+04	3.346360E+01
83.7	5.761406E+02	1.930277E+04	3.350358E+01
83.8	5.771732E+02	1.936044E+04	3.354355E+01
83.9	5.782064E+02	1.941821E+04	3.358352E+01
84.0	5.792402E+02	1.947608E+04	3.362349E+01
84.1	5.802746E+02	1.953405E+04	3.366346E+01
84.2	5.813097E+02	1.959213E+04	3.370344E+01
84.3	5.823453E+02	1.965032E+04	3.374341E+01
84.4	5.833816E+02	1.970860E+04	3.378338E+01
84.5	5.844185E+02	1.976699E+04	3.382335E+01
84.6	5.854560E+02	1.982549E+04	3.386332E+01
84.7	5.864941E+02	1.988408E+04	3.390330E+01
84.8	5.875328E+02	1.994278E+04	3.394327E+01
84.9	5.885721E+02	2.000159E+04	3.398324E+01
85.0	5.896121E+02	2.006050E+04	3.402321E+01

$\alpha = \mu/kT$	$U_{1/2}(\alpha)$	$U_{3/2}(\alpha)$	$U_{3/2}(\alpha)/U_{1/2}(\alpha)$
85.1	5.906526E+02	2.011951E+04	3.406319E+01
85.2	[illegible]	2.017863E+04	3.410316E+01
85.3	5.927356E+02	2.023785E+04	3.414313E+01
85.4	5.937780E+02	2.029718E+04	3.418311E+01
85.5	5.948210E+02	2.035661E+04	3.422308E+01
85.6	5.958646E+02	2.041614E+04	3.426305E+01
85.7	5.969088E+02	2.047578E+04	3.430303E+01
85.8	5.979537E+02	2.053552E+04	3.434300E+01
85.9	5.989991E+02	2.059537E+04	3.438297E+01
86.0	6.000452E+02	2.065532E+04	3.442295E+01
86.1	6.010918E+02	2.071538E+04	3.446292E+01
86.2	6.021391E+02	2.077554E+04	3.450289E+01
86.3	6.031870E+02	2.083581E+04	3.454287E+01
86.4	6.042355E+02	2.089618E+04	3.458284E+01
86.5	6.052846E+02	2.095665E+04	3.462281E+01
86.6	6.063343E+02	2.101723E+04	3.466279E+01
86.7	6.073846E+02	2.107792E+04	3.470276E+01
86.8	6.084355E+02	2.113871E+04	3.474273E+01
86.9	6.094870E+02	2.119961E+04	3.478271E+01
87.0	6.105391E+02	2.126061E+04	3.482268E+01
87.1	6.115918E+02	2.132172E+04	3.486266E+01
87.2	6.126452E+02	2.138293E+04	3.490263E+01
87.3	6.136991E+02	2.144424E+04	3.494260E+01
87.4	6.147536E+02	2.150567E+04	3.498258E+01
87.5	6.158088E+02	2.156720E+04	3.502255E+01
87.6	6.168645E+02	2.162883E+04	3.506253E+01
87.7	6.179209E+02	2.169057E+04	3.510250E+01
87.8	6.189778E+02	2.175241E+04	3.514247E+01
87.9	6.200354E+02	2.181436E+04	3.518245E+01
88.0	6.210936E+02	2.187642E+04	3.522242E+01
88.1	6.221523E+02	2.193858E+04	3.526240E+01
88.2	6.232117E+02	2.200085E+04	3.530237E+01
88.3	6.242716E+02	2.206322E+04	3.534235E+01
88.4	6.253322E+02	2.212571E+04	3.538232E+01
88.5	6.263933E+02	2.218829E+04	3.542230E+01
88.6	6.274551E+02	2.225098E+04	3.546227E+01
88.7	6.285175E+02	2.231378E+04	3.550225E+01
88.8	6.295804E+02	2.237669E+04	3.554222E+01
88.9	6.306440E+02	2.243970E+04	3.558220E+01
89.0	6.317081E+02	2.250282E+04	3.562217E+01
89.1	6.327729E+02	2.256604E+04	3.566215E+01
89.2	6.338383E+02	2.262937E+04	3.570212E+01
89.3	6.349042E+02	2.269281E+04	3.574210E+01
89.4	6.359707E+02	2.275635E+04	3.578207E+01
89.5	6.370379E+02	2.282000E+04	3.582205E+01
89.6	6.381056E+02	2.288376E+04	3.586202E+01
89.7	6.391740E+02	2.294762E+04	3.590200E+01
89.8	6.402429E+02	2.301159E+04	3.594197E+01
89.9	6.413124E+02	2.307567E+04	3.598195E+01
90.0	6.423826E+02	2.313986E+04	3.602193E+01

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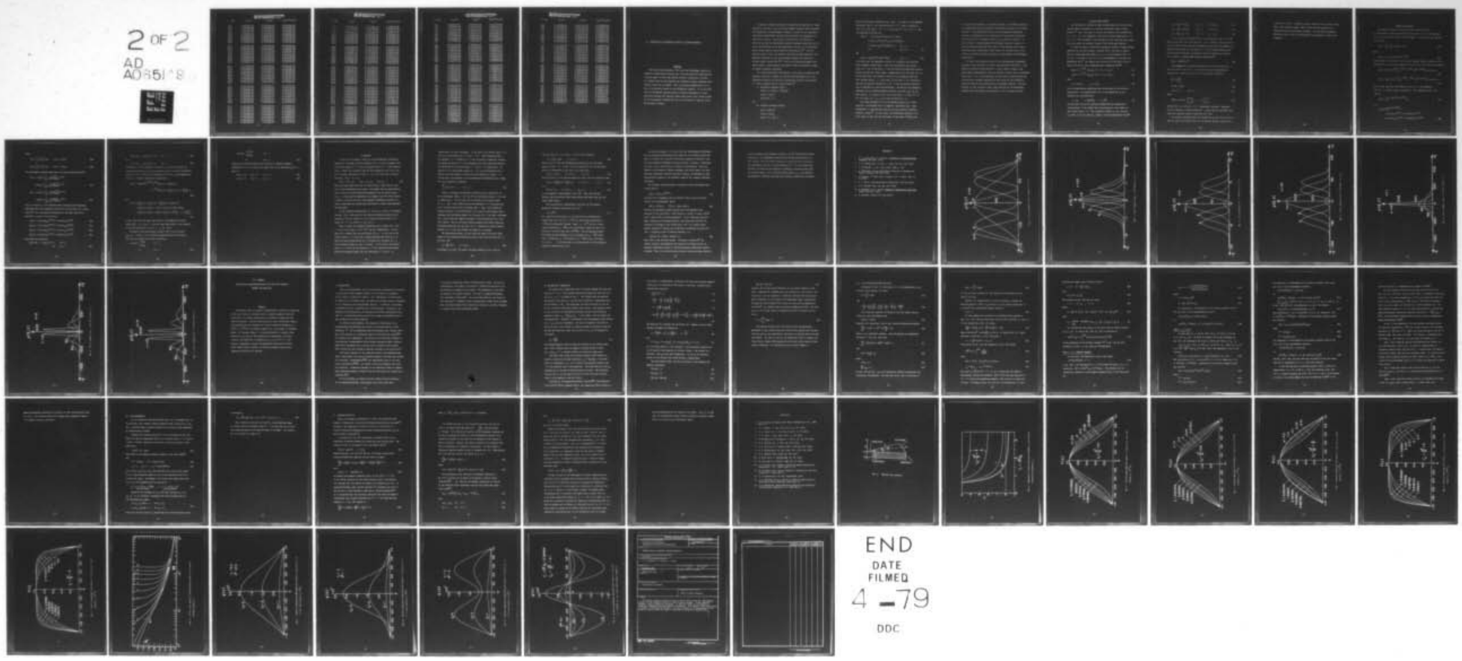

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$\alpha = \mu/kT$	$U_{1/2}(\alpha)$	$U_{3/2}(\alpha)$	$U_{3/2}(\alpha)/U_{1/2}(\alpha)$
90.1	6.434533E+02	2.320415E+04	3.606190E+01
90.2	6.445246E+02	2.326855E+04	3.610188E+01
90.3	6.455965E+02	2.333305E+04	3.614185E+01
90.4	6.466690E+02	2.339767E+04	3.618183E+01
90.5	6.477421E+02	2.346239E+04	3.622180E+01
90.6	6.488158E+02	2.352721E+04	3.626178E+01
90.7	6.498900E+02	2.359215E+04	3.630176E+01
90.8	6.509649E+02	2.365719E+04	3.634173E+01
90.9	6.520404E+02	2.372234E+04	3.638171E+01
91.0	6.531164E+02	2.378760E+04	3.642168E+01
91.1	6.541931E+02	2.385297E+04	3.646166E+01
91.2	6.552703E+02	2.391844E+04	3.650164E+01
91.3	6.563482E+02	2.398402E+04	3.654161E+01
91.4	6.574266E+02	2.404971E+04	3.658159E+01
91.5	6.585056E+02	2.411551E+04	3.662157E+01
91.6	6.595852E+02	2.418141E+04	3.666154E+01
91.7	6.606654E+02	2.424742E+04	3.670152E+01
91.8	6.617462E+02	2.431354E+04	3.674150E+01
91.9	6.628275E+02	2.437977E+04	3.678147E+01
92.0	6.639095E+02	2.444611E+04	3.682145E+01
92.1	6.649920E+02	2.451255E+04	3.686143E+01
92.2	6.660752E+02	2.457911E+04	3.690140E+01
92.3	6.671589E+02	2.464577E+04	3.694138E+01
92.4	6.682432E+02	2.471254E+04	3.698136E+01
92.5	6.693281E+02	2.477942E+04	3.702133E+01
92.6	6.704136E+02	2.484640E+04	3.706131E+01
92.7	6.714996E+02	2.491350E+04	3.710129E+01
92.8	6.725863E+02	2.498070E+04	3.714126E+01
92.9	6.736735E+02	2.504802E+04	3.718124E+01
93.0	6.747613E+02	2.511544E+04	3.722122E+01
93.1	6.758498E+02	2.518297E+04	3.726120E+01
93.2	6.769387E+02	2.525061E+04	3.730117E+01
93.3	6.780283E+02	2.531836E+04	3.734115E+01
93.4	6.791185E+02	2.538622E+04	3.738113E+01
93.5	6.802092E+02	2.545418E+04	3.742111E+01
93.6	6.813006E+02	2.552226E+04	3.746108E+01
93.7	6.823925E+02	2.559044E+04	3.750106E+01
93.8	6.834850E+02	2.565874E+04	3.754104E+01
93.9	6.845781E+02	2.572714E+04	3.758102E+01
94.0	6.856717E+02	2.579565E+04	3.762099E+01
94.1	6.867660E+02	2.586427E+04	3.766097E+01
94.2	6.878608E+02	2.593300E+04	3.770095E+01
94.3	6.889562E+02	2.600185E+04	3.774093E+01
94.4	6.900522E+02	2.607080E+04	3.778090E+01
94.5	6.911488E+02	2.613986E+04	3.782088E+01
94.6	6.922459E+02	2.620903E+04	3.786086E+01
94.7	6.933436E+02	2.627830E+04	3.790084E+01
94.8	6.944419E+02	2.634769E+04	3.794082E+01
94.9	6.955408E+02	2.641719E+04	3.798079E+01
95.0	6.966403E+02	2.648680E+04	3.802077E+01

$\alpha = \mu/kT$	$U_{1/2}(\alpha)$	$U_{3/2}(\alpha)$	$U_{3/2}(\alpha)/U_{1/2}(\alpha)$
95.1	6.977404E+02	2.655652E+04	3.806075E+01
95.2	6.988410E+02	2.662655E+04	3.810073E+01
95.3	6.999422E+02	2.669629E+04	3.814071E+01
95.4	7.010440E+02	2.676634E+04	3.818068E+01
95.5	7.021463E+02	2.683650E+04	3.822066E+01
95.6	7.032493E+02	2.690677E+04	3.826064E+01
95.7	7.043528E+02	2.697715E+04	3.830062E+01
95.8	7.054569E+02	2.704764E+04	3.834060E+01
95.9	7.065615E+02	2.711824E+04	3.838058E+01
96.0	7.076668E+02	2.718895E+04	3.842056E+01
96.1	7.087726E+02	2.725977E+04	3.846053E+01
96.2	7.098790E+02	2.733071E+04	3.850051E+01
96.3	7.109860E+02	2.740175E+04	3.854049E+01
96.4	7.120935E+02	2.747290E+04	3.858047E+01
96.5	7.132016E+02	2.754417E+04	3.862045E+01
96.6	7.143103E+02	2.761554E+04	3.866043E+01
96.7	7.154196E+02	2.768703E+04	3.870041E+01
96.8	7.165294E+02	2.775863E+04	3.874039E+01
96.9	7.176399E+02	2.783034E+04	3.878036E+01
97.0	7.187508E+02	2.790215E+04	3.882034E+01
97.1	7.198624E+02	2.797409E+04	3.886032E+01
97.2	7.209745E+02	2.804613E+04	3.890030E+01
97.3	7.220873E+02	2.811828E+04	3.894028E+01
97.4	7.232005E+02	2.819054E+04	3.898026E+01
97.5	7.243144E+02	2.826292E+04	3.902024E+01
97.6	7.254288E+02	2.833541E+04	3.906022E+01
97.7	7.265438E+02	2.840801E+04	3.910020E+01
97.8	7.276594E+02	2.848072E+04	3.914018E+01
97.9	7.287755E+02	2.855354E+04	3.918016E+01
98.0	7.298922E+02	2.862647E+04	3.922014E+01
98.1	7.310095E+02	2.869952E+04	3.926012E+01
98.2	7.321273E+02	2.877267E+04	3.930010E+01
98.3	7.332457E+02	2.884594E+04	3.934007E+01
98.4	7.343647E+02	2.891932E+04	3.938005E+01
98.5	7.354843E+02	2.899282E+04	3.942003E+01
98.6	7.366044E+02	2.906642E+04	3.946001E+01
98.7	7.377251E+02	2.914014E+04	3.949999E+01
98.8	7.388463E+02	2.921396E+04	3.953997E+01
98.9	7.399682E+02	2.928791E+04	3.957995E+01
99.0	7.410906E+02	2.936196E+04	3.961993E+01
99.1	7.422135E+02	2.943612E+04	3.965991E+01
99.2	7.433370E+02	2.951040E+04	3.969989E+01
99.3	7.444611E+02	2.958479E+04	3.973987E+01
99.4	7.455858E+02	2.965929E+04	3.977985E+01
99.5	7.467110E+02	2.973391E+04	3.981983E+01
99.6	7.478368E+02	2.980864E+04	3.985981E+01
99.7	7.489632E+02	2.988348E+04	3.989979E+01
99.8	7.500901E+02	2.995843E+04	3.993977E+01
99.9	7.512176E+02	3.003349E+04	3.997975E+01
100.0	7.523456E+02	3.010867E+04	4.001973E+01

$\alpha = \mu/kT$	$U_{1/2}(\alpha)$	$U_{3/2}(\alpha)$	$U_{3/2}(\alpha)/U_{1/2}(\alpha)$
110	8.679548E+02	3.820558E+04	4.401794E+01
120	9.889467E+02	4.748571E+04	4.801644E+01
130	1.115092E+03	5.800169E+04	5.201518E+01
140	1.246187E+03	6.980404E+04	5.601410E+01
150	1.382052E+03	8.294133E+04	6.001316E+01
160	1.522526E+03	9.746044E+04	6.401233E+01
170	1.667461E+03	1.134067E+05	6.801161E+01
180	1.816725E+03	1.308241E+05	7.201096E+01
190	1.970194E+03	1.497552E+05	7.601038E+01
200	2.127758E+03	1.702416E+05	8.000986E+01
210	2.289312E+03	1.923237E+05	8.400939E+01
220	2.454759E+03	2.160408E+05	8.800897E+01
230	2.624011E+03	2.414315E+05	9.200858E+01
240	2.796984E+03	2.685335E+05	9.600822E+01
250	2.973599E+03	2.973834E+05	1.000079E+02
260	3.153783E+03	3.280173E+05	1.040076E+02
270	3.337466E+03	3.604707E+05	1.080073E+02
280	3.524583E+03	3.947781E+05	1.120070E+02
290	3.715072E+03	4.309736E+05	1.160068E+02
300	3.908874E+03	4.690906E+05	1.200066E+02
310	4.105934E+03	5.091619E+05	1.240064E+02
320	4.306198E+03	5.512199E+05	1.280062E+02
330	4.509617E+03	5.952964E+05	1.320060E+02
340	4.716142E+03	6.414226E+05	1.360058E+02
350	4.925726E+03	6.896294E+05	1.400056E+02
360	5.138327E+03	7.399472E+05	1.440055E+02
370	5.353901E+03	7.924058E+05	1.480053E+02
380	5.572408E+03	8.470350E+05	1.520052E+02
390	5.793810E+03	9.038636E+05	1.560051E+02
400	6.018069E+03	9.629207E+05	1.600049E+02
410	6.245149E+03	1.024234E+06	1.640048E+02
420	6.475015E+03	1.087833E+06	1.680047E+02
430	6.707635E+03	1.153744E+06	1.720046E+02
440	6.942975E+03	1.221995E+06	1.760045E+02
450	7.181005E+03	1.292612E+06	1.800044E+02
460	7.421695E+03	1.365624E+06	1.840043E+02
470	7.665015E+03	1.441055E+06	1.880042E+02
480	7.910938E+03	1.518933E+06	1.920041E+02
490	8.159437E+03	1.599282E+06	1.960040E+02
500	8.410484E+03	1.682130E+06	2.000039E+02
510	8.664054E+03	1.767500E+06	2.040039E+02
520	8.920123E+03	1.855419E+06	2.080038E+02
530	9.178665E+03	1.945911E+06	2.120037E+02
540	9.439659E+03	2.039001E+06	2.160036E+02
550	9.703080E+03	2.134712E+06	2.200036E+02
560	9.968908E+03	2.233070E+06	2.240035E+02
570	1.023712E+04	2.334098E+06	2.280035E+02
580	1.050769E+04	2.437821E+06	2.320034E+02
590	1.078061E+04	2.544260E+06	2.360033E+02
600	1.105585E+04	2.653441E+06	2.400033E+02

$\alpha = \mu/kT$	$U_{1/2}(\alpha)$	$U_{3/2}(\alpha)$	$U_{3/2}(\alpha)/U_{1/2}(\alpha)$
610	1.133339E+04	2.765385E+06	2.440032E+02
620	1.161322E+04	2.880116E+06	2.480032E+02
630	1.189532E+04	2.997657E+06	2.520031E+02
640	1.217966E+04	3.118030E+06	2.560031E+02
650	1.246623E+04	3.241257E+06	2.600030E+02
660	1.275501E+04	3.367362E+06	2.640030E+02
670	1.304600E+04	3.496365E+06	2.680029E+02
680	1.333916E+04	3.628289E+06	2.720029E+02
690	1.363448E+04	3.763155E+06	2.760028E+02
700	1.393195E+04	3.900986E+06	2.800028E+02
710	1.423156E+04	4.041801E+06	2.840028E+02
720	1.453328E+04	4.185624E+06	2.880027E+02
730	1.483710E+04	4.332474E+06	2.920027E+02
740	1.514302E+04	4.482373E+06	2.960027E+02
750	1.545100E+04	4.635341E+06	3.000026E+02
760	1.576105E+04	4.791400E+06	3.040026E+02
770	1.607314E+04	4.950569E+06	3.080025E+02
780	1.638727E+04	5.112869E+06	3.120025E+02
790	1.670342E+04	5.278321E+06	3.160025E+02
800	1.702157E+04	5.446944E+06	3.200025E+02
810	1.734172E+04	5.618759E+06	3.240024E+02
820	1.766385E+04	5.793785E+06	3.280024E+02
830	1.798795E+04	5.972043E+06	3.320024E+02
840	1.831401E+04	6.153551E+06	3.360023E+02
850	1.864202E+04	6.338330E+06	3.400023E+02
860	1.897196E+04	6.526398E+06	3.440023E+02
870	1.930383E+04	6.717775E+06	3.480023E+02
880	1.963761E+04	6.912481E+06	3.520022E+02
890	1.997329E+04	7.110534E+06	3.560022E+02
900	2.031086E+04	7.311953E+06	3.600022E+02
910	2.065031E+04	7.516757E+06	3.640021E+02
920	2.099163E+04	7.724965E+06	3.680021E+02
930	2.133481E+04	7.936596E+06	3.720021E+02
940	2.167985E+04	8.151667E+06	3.760021E+02
950	2.202672E+04	8.370199E+06	3.800021E+02
960	2.237542E+04	8.592208E+06	3.840020E+02
970	2.272595E+04	8.817713E+06	3.880020E+02
980	2.307828E+04	9.046733E+06	3.920020E+02
990	2.343242E+04	9.279285E+06	3.960020E+02
1000	2.378835E+04	9.515387E+06	4.000020E+02

VI. PROPAGATION OF PROBABILITY DENSITY IN QUANTUM MECHANICS

Abstract

The initial-value problem of the force-free Schrödinger equation is solved in closed-form for initial wave functions which are constrained to a finite region of space and improper boundary conditions at infinity. It is shown that the wave function and probability density propagate with infinite speed into the space. This is explained mathematically to be due to the parabolic nature of the Schrödinger equation. It is concluded that the Schrödinger equation gives an adequate description of quasi-stationary systems with discrete energy eigen-values (true steady-systems in the hydrodynamic formulation), but is insufficient to describe proper time-dependent systems.

In general, transient processes are described by hyperbolic or quasi-hyperbolic equations, e.g., the (linear) Maxwell equations which determine the propagation of electromagnetic signals in vacuum or the (nonlinear) compressible hydrodynamic equations which determine the propagation of shock waves in gases. The parabolic equations for the diffusion of particles and heat are approximate equations assuming proportionality of fluxes and diffusion forces (disregarding the inertia of the molecules) which exhibit physical and mathematical deficiencies.[1,2] From the physical point of view, the essential difference between hyperbolic and parabolic equations is that they propagate signals with finite and infinite speed, respectively.[1,2] Since real processes propagate with finite speed, a proper physical description cannot be achieved by means of (quasi-) parabolic equations.

As a brief quantitative illustration to the nature of hyperbolic and parabolic solutions, consider the initial-value problems for the one-dimensional diffusion of particles concentrated initially in the plane $x = 0$ with a density $n(x,0) = \bar{n}_o\delta(x)$ based on[2]

i) Hyperbolic Diffusion Theory:

$$\partial^2 n/\partial t^2 + \tau^{-1}\partial n/\partial t = c^2\partial^2 n/\partial x^2 \quad ,$$

$$n(x,0) = \bar{n}_o\delta(x) \quad ,$$

$$\partial n(x,0)/\partial t = 0 \quad ,$$

and

ii) Parabolic Diffusion Theory:

$$\partial n/\partial t = D\partial^2 n/\partial x^2 \quad ,$$

$$n(x,0) = \bar{n}_o\delta(x) \quad ,$$

$$n(x,t) \to 0, \quad |x| \to \infty \quad ,$$

where the diffusion coefficient (D), speed of sound (c), and momentum relaxation time (τ) are interrelated by $D = c^2\tau$. Case i) reduces to case ii) in the limit $\tau \to 0$, $c \to \infty$, such that $c^2\tau \to D$, $0 < D < \infty$. The corresponding solutions are:

$$\begin{aligned} n(x,t) = {} & \tfrac{1}{2}\,\bar{n}_o \exp(-t/2\tau)\{\delta(x-ct) + \delta(x+ct) \\ & + (t/2\tau)\,[c^2t^2-x^2]^{-\frac{1}{2}}\, I_1([c^2t^2-x^2]^{\frac{1}{2}}/2c\tau) \\ & + (1/2c\tau)\, I_o([c^2t^2-x^2]^{\frac{1}{2}}/2c\tau)\}\ , \quad |x| < ct\ , \\ = {} & 0 \qquad\qquad\qquad\qquad\qquad\qquad\quad , \quad |x| > ct\ , \end{aligned} \tag{i}$$

and

$$n(x,t) = \bar{n}_o(4\pi Dt)^{-\frac{1}{2}} \exp(-x^2/4Dt) \qquad , \qquad |x| \le \infty. \tag{ii}$$

It is seen that the hyperbolic solution (i) propagates the particles with a maximum, finite speed c such that discontinuous wave fronts exist at $x = \pm ct$ at any time $t < \infty$ and the initial discontinuity $n(x,0) = \bar{n}_o\delta(x)$ is moved with finite speed c symmetrically into the space $|x| < \infty$. On the other hand, the parabolic solution (ii) propagates particles with infinite speed, $c = \infty$, such that, at any time $t > 0$, a continuous particle distribution extends up to $|x| = \infty$ and the initial discontinuity is smoothed out quasi-instantaneously. By means of the asymptotic expansions for the modified Bessel functions $I_o(x)$ and $I_1(x)$ one can show that Eq. (i) reduces to Eq. (ii) in the limit $\tau \to 0$ with $c^2\tau \to D$ for all points $|x| < ct$, i.e., not for the space $|x| > ct$.

The common designation of the Schrödinger equation as a "wave equation" is misleading since it suggests a hyperbolic type. Even Schrödinger, in comparing his equation with the parabolic diffusion equation, states:[3] "In both cases, the differential equation is of first order in time, but the occurrence of the factor $\sqrt{-1}$ gives the

wave equation a hyperbolic, or physically spoken, a reversible character, as distinguished from the parabolic-irreversible character of the Fokker equation." The mathematical type of the time-dependent Schrödinger equation is determined by the coefficients of the spatial derivatives of second order[4], according to which it is a parabolic equation. A first attempt at hyperbolizing the Schrödinger equation by means of a pure mathematical approach[5] has lead to a wave equation with complex, energy-dependent coefficients, which reduces in limiting cases to the Lorentz and Galilei covariant equations of Klein-Gordon and Schrödinger, respectively.

In view of the parabolic nature of the time-dependent Schrödinger equation, "wave" functions $\psi = \psi(\vec{r},t)$ and probability densities $\rho = \psi\psi^*$, which are initially (t=0) limited to a finite region in space, will spread quasi-instantaneously over the infinity of space after an arbitrary short time $t > 0$. In the following, we will discuss quantitatively the time and space dependent spreading of wave functions and probability densities based on the nonrelativistic Schrödinger equation. For this purpose, we will evaluate a basic Cauchy problem for the Schrödinger equation with the initial value limited to a finite region in space.

INITIAL-VALUE PROBLEM

The spreading of ψ-waves has been analyzed based on an initial wave function which corresponds to a Gaussian probability density ("wave packet").[6] Since this type of initial distribution (t=0) extends from $x = -\infty$ to $x = +\infty$, all subsequent ($t>0$) wave functions extend from $x = -\infty$ to $x = +\infty$. This treatment of the spreading of wave functions does, therefore, not expose the parabolic nature of the Schrödinger equation.

In order to deduce a physically realizable, spatially limited initial condition for the wave function, consider first a particle of mass m which is constrained to the region $-a \leq x \leq +a$ with potential $V(x) = 0$ for $|x| < a$ and $V(x) = \infty$ for $x = \pm a \pm 0$, corresponding to the ideal one-dimensional box.[6] The complex wave functions for the state n of the particle in the box are in dimensionless, normalized form:[6]

$$\Psi_n(\xi,\tau) = e^{-i\varepsilon_n\tau} \cos \alpha_n\xi, \quad |\xi| < 1, \quad n = 1,3,5... \quad , \tag{1}$$

$$\Psi_n(\xi,\tau) = e^{-i\varepsilon_n\tau} \sin \alpha_n\xi, \quad |\xi| < 1, \quad n = 2,4,6,... \quad , \tag{2}$$

where

$$\alpha_n = (\pi/2)n, \qquad \varepsilon_n = \tfrac{1}{2}\alpha_n^2 \quad , \tag{3}$$

are the dimensionless eigenvalues and energy levels of the particle, respectively. The dimensionless (Ψ,ξ,τ) and dimensional (ψ,x,t) variables are interrelated by

$$\xi = x/a, \qquad \tau = t/(ma^2/\hbar), \qquad \Psi = a^{\frac{1}{2}}\psi \quad . \tag{4}$$

The solutions (1) and (2) represent symmetrical and asymmetrical "oscillations" of the complex wave function $\Psi_n(\xi,\tau)$ with respect to the central plane $\xi = 0$. The probability density ρ_n and velocity $\vec{v}_n$ fields of the box particle, however, are time-independent fields[7]

$$\rho_n = \Psi_n \Psi_n^* = \cos^2 \alpha_n \xi, \qquad |\xi| < 1, \qquad n = 1,3,5,\ldots \quad , \tag{5}$$

$$\rho_n = \Psi_n \Psi_n^* = \sin^2 \alpha_n \xi, \qquad |\xi| < 1, \qquad n = 2,4,6,\ldots \quad , \tag{6}$$

$$\vec{v}_n = (\hbar/m)\, \nabla\Phi_n = \vec{0}, \qquad |\xi| < 1, \qquad \Phi_n = -i\varepsilon_n \tau \quad . \tag{7}$$

If the potential walls $V(\xi=\pm1\pm0) = \infty$ are removed at time $\tau = 0$, the particle will move out of the box $|\xi| < 1$, i.e., either to the left ($\xi<-1$) or to the right ($\xi>+1$) with equal probability, since the box geometry is symmetric with respect to the plane $\xi = 0$. This follows directly from the uncertainty principle, according to which the particle in the box has in state n a (dimensional) momentum uncertainty[7)]

$$\pm\Delta p_n = \pm\tfrac{1}{2}\hbar(n\pi/2)a^{-1} \quad . \tag{8}$$

The propagation of the wave function $\Psi(\xi,\tau)$ out of the box $|\xi| < 1$ into the infinite space $|\xi| \leq \infty$ is described by the (dimensionless) initial-value problem for the force-free Schrödinger equation:

$$\frac{\partial\Psi}{\partial\tau} = \frac{i}{2}\frac{\partial^2\Psi}{\partial\xi^2} \quad , \tag{9}$$

$$\Psi(\xi,0) = \Psi_n^o(\xi) \quad , \tag{10}$$

$$\Psi(\xi,\tau) \to 0, \qquad |\xi| \to \infty \quad , \tag{11}$$

where

$$\Psi_n^o(\xi) = H(1-|\xi|) \begin{matrix} \cos\alpha_n\xi \\ \sin\alpha_n\xi \end{matrix} , \quad n = \begin{matrix} 1,3,5\ldots \\ 2,4,6\ldots \end{matrix} \tag{12}$$

and $H(1-|\xi|) = 1;\ 0$ for $|\xi| < 1;\ > 1$ (Heaviside function). Equations (9)-(11) represent a Cauchy problem for proper initial conditions (10), since only improper boundary conditions (11) exist.

The initial conditions have been assumed in the form of Eq. (12), in order to relate the initial state to the familiar physical situation of

a particle in a box. It should be noted, however, that an initial value $\psi_n^o(\xi)$ could have been chosen, which is any arbitrary function of ξ, compatible with the uncertainty principle. For any initial condition different from Eq. (12), the calculations are analagous to those to be presented.

ANALYTICAL SOLUTION

The general solution of the initial-value problem (9)-(11) is represented by a Fourier integral for the space $|\xi| \leq \infty$. This approach gives, for an arbitrary initial condition $\Psi_n^o(\xi)$, a solution of Eqs. (9)-(11) in the form:

$$\Psi(\xi,\tau) = \int_{-\infty}^{+\infty} \Psi_n^o(\xi')\, G(\xi,\xi',\tau)\,d\xi' \qquad (13)$$

where

$$G(\xi,\xi',\tau) = (-i/2\pi\tau)^{\frac{1}{2}}\, e^{i(\xi'-\xi)^2/2\tau} \qquad (14)$$

is the Green's function of Eq. (9) for the improper boundary conditions (11). Substitution of the initial conditions (12) into Eq. (13) yields the solutions

$$\Psi(\xi,\tau) = \frac{1}{2}(-i/2\pi\tau)^{\frac{1}{2}} \int_{-1}^{+1} (e^{+i\alpha_n\xi'} + e^{-i\alpha_n\xi'})\, e^{i(\xi'-\xi)^2/2\tau}\, d\xi', n=1,3,5..., \qquad (15)$$

$$\Psi(\xi,\tau) = \frac{1}{2i}\, (-i/2\pi\tau)^{\frac{1}{2}} \int_{-1}^{+1} (e^{+i\alpha_n\xi'} - e^{-i\alpha_n\xi'})\, e^{i(\xi'-\xi)^2/2\tau}\, d\xi', n=2,4,6..., \qquad (16)$$

for the wave functions with symmetric (n=1,3,5,...) and asymmetric (n=2,4,6...) initial values, respectively. The integrals in Eqs. (15)-(16) are

$$J_n^{\pm}(\xi) = \int_{-1}^{+1} e^{\pm i\alpha_n\xi' + i(\xi'-\xi)^2/2\tau}\, d\xi' \quad , \qquad (17)$$

i.e.,

$$J_n^{\pm}(\xi)/[\pi^{\frac{1}{2}}\tau^{\frac{1}{2}} e^{\pm i\alpha_n\xi - i\frac{1}{2}\alpha_n^2\tau}] =$$

$$C([1-\xi\pm\alpha_n\tau]\pi^{-\frac{1}{2}}\tau^{-\frac{1}{2}}) + C([1+\xi\mp\alpha_n\tau]\pi^{-\frac{1}{2}}\tau^{-\frac{1}{2}})$$

$$+ i\{S([1-\xi\pm\alpha_n\tau]\pi^{-\frac{1}{2}}\tau^{-\frac{1}{2}}) + S([1+\xi\mp\alpha_n\tau]\pi^{-\frac{1}{2}}\tau^{-\frac{1}{2}})\} \quad , \qquad (18)$$

where

$$C(z) = \int_0^z \cos\frac{\pi}{2}x^2\,dx, \qquad C(-z) = -C(+z) \quad , \tag{19}$$

$$S(z) = \int_0^z \sin\frac{\pi}{2}x^2\,dx, \qquad S(-z) = -S(+z) \quad , \tag{20}$$

are the Fresnel integrals which have the series representations,[8)]

$$C(z) = \cos(\frac{\pi}{2}z^2)\sum_{\nu=0}^{\infty}\frac{(-1)^\nu \pi^{2\nu}}{1\cdot 3 \ldots (4\nu+1)} z^{4\nu+1}$$

$$+\sin(\frac{\pi}{2}z^2)\sum_{\nu=0}^{\infty}\frac{(-1)^\nu \pi^{2\nu+1}}{1\cdot 3 \ldots (4\nu+3)} z^{4\nu+3} \quad , \tag{21}$$

$$S(z) = -\cos(\frac{\pi}{2}z^2)\sum_{\nu=0}^{\infty}\frac{(-1)^\nu \pi^{2\nu+1}}{1\cdot 3 \ldots (4\nu+3)} z^{4\nu+3}$$

$$+\sin(\frac{\pi}{2}z^2)\sum_{\nu=0}^{\infty}\frac{(-1)^\nu \pi^{2\nu}}{1\cdot 3 \ldots (4\nu+1)} z^{4\nu+1} \quad . \tag{22}$$

$C(z) \geq 0$ and $S(z) \geq 0$ are functions which oscillate with decreasing amplitude about their asymptotic values $C(\infty) = \frac{1}{2}$ and $S(\infty) = \frac{1}{2}$, respectively.[8)] For a convenient representation, the final results are expressed in terms of the functions,

$$P_n(\xi,\tau) = C([1-\xi+\alpha_n\tau]\pi^{-\frac{1}{2}}\tau^{-\frac{1}{2}}) + C([1+\xi-\alpha_n\tau]\pi^{-\frac{1}{2}}\tau^{-\frac{1}{2}}) \quad , \tag{23}$$

$$Q_n(\xi,\tau) = C([1-\xi-\alpha_n\tau]\pi^{-\frac{1}{2}}\tau^{-\frac{1}{2}}) + C([1+\xi+\alpha_n\tau]\pi^{-\frac{1}{2}}\tau^{-\frac{1}{2}}) \quad , \tag{24}$$

$$M_n(\xi,\tau) = S([1-\xi+\alpha_n\tau]\pi^{-\frac{1}{2}}\tau^{-\frac{1}{2}}) + S([1+\xi-\alpha_n\tau]\pi^{-\frac{1}{2}}\tau^{-\frac{1}{2}}) \quad , \tag{25}$$

$$N_n(\xi,\tau) = S([1-\xi-\alpha_n\tau]\pi^{-\frac{1}{2}}\tau^{-\frac{1}{2}}) + S([1+\xi+\alpha_n\tau]\pi^{-\frac{1}{2}}\tau^{-\frac{1}{2}}) \quad , \tag{26}$$

which have the properties,

$$P_n(\xi,\tau=0),\ldots, N_n(\xi,\tau=0) = 1, \qquad |\xi| < 1 \quad ,$$

$$= 0, \qquad |\xi| > 1 \quad , \tag{27}$$

$$P_n(\xi=\infty,\tau),\ldots,\ N_n(\xi=\infty,\tau) = 0, \qquad 0 \leq \tau < \infty \qquad , \tag{28}$$

and

$$P_n(\xi,\tau>0),\ldots,\ N_n(\xi,\tau>0) > 0, \qquad |\xi| < \infty \qquad . \tag{29}$$

Substitution of the integrals (18) into Eqs. (15) and (16) gives the closed-form solutions for the wave functions $\Psi(\xi,\tau)$ and their probability densities $\rho(\xi,\tau) = \Psi\Psi^*$ for the symmetrical (n=1,3,5...) and asymmetrical (n=2,4,6...) initial conditions, respectively:

$$\Psi(\xi,\tau) = \tfrac{1}{2}(\mp i/2)^{\frac{1}{2}} e^{-i\frac{1}{2}\alpha_n^2\tau} \{e^{+i\alpha_n\xi}[P_n(\xi,\tau) + iM_n(\xi,\tau)]$$

$$\pm e^{-i\alpha_n\xi}[Q_n(\xi,\tau) + iN_n(\xi,\tau)]\}, \quad n = \begin{matrix}1,3,5,\ldots\\2,4,6,\ldots\end{matrix} \tag{30}$$

and

$$\rho(\xi,\tau) = \frac{1}{8}[P_n(\xi,\tau) \mp Q_n(\xi,\tau)]^2 + \frac{1}{8}[M_n(\xi,\tau) \mp N_n(\xi,\tau)]^2$$

$$\pm \tfrac{1}{2}[P_n(\xi,\tau)\, N_n(\xi,\tau) - Q_n(\xi,\tau)\, M_n(\xi,\tau)] \sin\alpha_n\xi \cos\alpha_n\xi$$

$$\pm \tfrac{1}{2}[P_n(\xi,\tau)\, Q_n(\xi,\tau) + M_n(\xi,\tau)\, N_n(\xi,\tau)] \cos^2\alpha_n\xi, \quad n = \begin{matrix}1,3,5,\ldots\\2,4,6,\ldots\end{matrix} \tag{31}$$

In Eqs. (30)-(31), the upper sign refers to the symmetrical initial values $\Psi_n^o(\xi)$, n = 1,3,5,..., and the lower sign refers to the asymmetrical initial values Ψ_n^o, n = 2,4,6,... [s. Eq. (12)].

By means of the discontinuous property (28) of the functions $P_n(\xi,\tau),\ldots,\ N_n(\xi,\tau)$, it is readily demonstrated that the solutions (30) and (31) satisfy the initial conditions,

$$\Psi(\xi,\tau=0) = \begin{matrix}\cos\alpha_n\xi\\ \sin\alpha_n\xi\end{matrix} \quad , \qquad |\xi| < 1 \qquad ,$$

$$= 0 \quad , \qquad |\xi| > 1 \qquad , \tag{32}$$

$$\rho(\xi,\tau=0) = \begin{matrix} \cos^2\alpha_n\xi \\ \sin^2\alpha_n\xi \end{matrix} \quad , \qquad |\xi| < 1 \quad ,$$

$$= 0 \quad , \qquad |\xi| > 1 \quad . \tag{33}$$

Finally, the solutions (30) and (31) satisfy the improper boundary conditions at $\xi = \infty$ in view of the limits (28) of the functions $P_n(\xi,\tau)$,..., $N_n(\xi,\tau)$,

$$\Psi(\xi,\tau) \to 0, \quad |\xi| \to \infty, \quad 0 \leq \tau < \infty \quad , \tag{34}$$

$$\rho(\xi,\tau) \to 0, \quad |\xi| \to \infty \quad 0 \leq \tau < \infty \quad . \tag{35}$$

DISCUSSION

In view of the parabolic nature of the time-dependent Schrödinger equation (9), we expect that the signals $\Psi(\xi,\tau)$ or $\rho(\xi,\tau)$ propagate from the initial region $|\xi| < 1$ into the adjacent space $|\xi| > 1$ with infinite speed. Indeed, the solutions (30) and (31) demonstrate that $\Psi(\xi,\tau)$ and $\rho(\xi,\tau)$ extend, continuously, from $\xi = -\infty$ to $\xi = +\infty$ after any, no matter how short, time $\tau > 0$, i.e.,

$$|\Psi(\xi,\tau > 0)| \geq 0, \qquad \rho(\xi,\tau > 0) \geq 0, \qquad \text{for } |\xi| < \infty \tag{36}$$

where the equal signs hold only for those points ξ where $|\Psi(\xi,\tau)|$ and $\rho(\xi,\tau)$ touch tangentially the ξ-axis. This means that the distributions $\Psi(\xi\ \tau)$ and $\rho(\xi,\tau)$ spread quasi-instantaneously over the entire space $|\xi| \leq \infty$. Since physical phenomena can propagate only with finite speed, it must be concluded that the time-dependent Schrödinger equation provides a misleading and insufficient description of proper time-dependent microprocesses.

In the following illustrations, it is $\rho(-\xi,\tau) = \rho(+\xi,\tau)$ for symmetry reasons. Fig. 1 gives plots of the initial probability density $\rho(\xi,0)$ versus $|\xi| \geq 0$ for the states $n = 1,2,3$, which is non-zero only in the region $|\xi| < 1$, i.e., $\rho(\xi,0) = 0$ for $|\xi| \geq 1$.

Figs. 2-4 give the probability densities $\rho(\xi,\tau)$ versus $|\xi| \geq 0$ for $n = 1,2,3$ at the times $\tau = 10^{-1}$, 10^{0}, and 10^{1}, respectively. In each case $\rho(\xi,\tau)$ extends over the entire space $|\xi| \leq \infty$. In the case of the initial state $n = 1$, $\rho(\xi,0)$ has a single maximum which stays at $\xi = 0$, while the entire distribution $\rho(\xi,\tau)$ diffuses into the space $|\xi| \leq \infty$ with decreasing height as time τ increases. In the case of the initial state $n = 2$, $\rho(\xi,0)$ has two maxima at $\xi = \pm 1/2$, which move with finite speed and decreasing height into the half-spaces $\xi < 0$ and $\xi > 0$,

respectively, as time τ increases. In the case of the initial state $n = 3$, $\rho(\xi,0)$ has three maxima at $\xi = 0$ and $\xi = \pm 2/3$. With increasing time τ, the maximum at $\xi = 0$ remains at $\xi = 0$ and decreases in amplitude, whereas the maxima initially at $\xi = \pm 2/3$ propagate with finite speed and decreasing amplitudes into the half-spaces $\xi < 0$ and $\xi > 0$, respectively. In addition to the main maxima present at $\tau \geq 0$, the distributions $\rho(\xi,\tau)$ exhibit secondary maxima of relatively small amplitude at times $\tau > 0$. In each case, the diffusion of the probability distributions $\rho(\xi,\tau)$ leaves the total probability conserved,

$$\int_{-\infty}^{+\infty} \rho(\xi,\tau)\, d\xi = 1, \qquad 0 \leq \tau \leq \infty. \tag{37}$$

Figs. 5-7 present the probability densities $\rho(\xi,\tau)$ versus $|\xi| \geq 0$ at the successive times $\tau = 10, 30, 50$ for the initial states $n = 1,2$, and 3, respectively. In each case, $\rho(\xi,\tau)$ extends over the entire space $|\xi| \leq \infty$. Fig. 5 (n=1) shows how the main maximum at $\xi = 0$ decreases and $\rho(\xi,\tau)$ becomes flatter with increasing time τ. Fig. 6 (n=2) shows the temporal decrease of the distribution $\rho(\xi,\tau)$ with two main maxima, which propagate with decreasing height into the space $|\xi| > 0$ as time τ increases. Fig. 6 (n=3) shows the temporal decrease of the distribution $\rho(\xi,\tau)$ with three maxima. The two off-center maxima propagate with finite speed and decreasing amplitude into the space $|\xi| \leq \infty$, whereas the central maximum remains at $\xi = 0$ and loses rapidly in height as τ increases.

The dimensionless speed U_n with which the (main) off-center ($\xi \neq 0$) maxima of the probability distribution $\rho(\xi,\tau)$ move into the space $|\xi| \leq \infty$ is by Eq. (31)

$$U_n = \frac{\Delta\xi}{\Delta\tau} = \frac{\pi}{2}\, n, \qquad n = 2,3,4,\ldots \quad . \tag{38}$$

Accordingly, at time τ the (main) off-center maxima of $\rho(\xi,\tau)$ are in

the half spaces $\xi < 0$ (-) and $\xi > 0$ (+) at the locations

$$\pm\hat{\xi} = \hat{\xi}_o(n) + \frac{\pi}{2}n\tau, \qquad n = 2,3,4,\ldots \qquad . \tag{39}$$

where $\pm\hat{\xi}_o(n) \neq 0$ are the dimensionless positions of the off-center maxima at time $\tau = 0$. By Eq. (31), the value $\hat{\rho}(\hat{\xi},\tau)$ of the off-center maxima is independent of the state n at large times,

$$\hat{\rho}(\hat{\xi},\tau) \cong 1/2\pi\tau, \qquad n = 2,3,4\ldots, \qquad \alpha_n\tau >> 1 \qquad . \tag{40}$$

The value $\hat{\rho}(0,\tau)$ of the central maxima is by Eq. (31) for arbitrary times,

$$\hat{\rho}(0,\tau) = \frac{1}{2}[P_n^2(0,\tau) + M_n^2(0,\tau)], \qquad n = 1,3,5\ldots, \qquad \tau \geq 0, \tag{41}$$

i.e.,

$$\hat{\rho}(0,\tau) \cong 0, \qquad n = 1,3,5\ldots, \qquad \alpha_n\tau >> 1 \qquad , \tag{42}$$

in the asymptotic approximation of Eq. (40). Eqs. (40) and (42) indicate that the central ($\hat{\xi}=0$) maxima decay much faster than the off-center ($\hat{\xi}\neq 0$) maxima.

The characteristic (dimensional) time scale of the quantum mechanical diffusion process is by Eq. (4)

$$t_o = ma^2/\hbar \qquad . \tag{43}$$

As a numerical illustration, it is noted that the characteristic length scale [Eq. (4)] is a ~ 10^{-8} cm for microscopic systems and a ~ 10^{0} cm for macroscopic systems. Hence, $t_o \sim 10^{-16}$ sec for a microscopic system and $t_o \sim 10^{0}$ sec for a macroscopic system in case of an electron or positron (m=9.108 x 10^{-28} gm). The corresponding speeds with which the main maxima of $\rho(\xi,\tau)$ propagate are $u_n \sim 10^{8}$n cm/sec for a = 10^{-8} cm and $u_n \sim 10^{0}$n cm/sec for a = 10^{0}cm by Eq. (38) where n = 2,3,4... . It is seen that u_n is nonrelativistic for microscopic β-particle systems with n ≤ 10.

As main conclusions, it is noted that the time-dependent Schrödinger equation spreads quantum-mechanical probability with infinite speed over space, as typical for a partial differential equation of parabolic type. The time-dependent Schrödinger equation provides, therefore, a physically not quite correct description of transient microprocesses. Since the maxima of the probability density propagate with finite speed, the time-dependent Schrödinger equation describes transient microsystems in some approximation (similar to the parabolic equation for classical diffusion processes).

By transient quantum systems or processes we mean those which have a wave function

$$\psi(\vec{r},t) = R(\vec{r},t)\, e^{i\Phi(\vec{r},t)} \tag{44}$$

of such $\vec{r}$ and t dependence that the density field ρ and the velocity field $\vec{v}$ are time-dependent, where

$$\rho(\vec{r},t) = R^2(\vec{r},t), \qquad \vec{v}(\vec{r},t) = (\hbar/m)\, \nabla\Phi(\vec{r},t) \tag{45}$$

Improper time-dependent quantum systems have time-dependent wave functions of the form $\psi(\vec{r},t) = R(\vec{r}) \exp[i\omega(t) + i\phi(\vec{r})]$ so that $\rho = R^2(\vec{r})$ and $\vec{v} = (\hbar/m)\, \nabla\phi(\vec{r})$ are time-independent. In the (nonlinear) hydrodynamic formulation of quantum mechanics, the fields $\rho(\vec{r})$ and $\vec{v}(\vec{r})$ are obtained as solutions of the steady-state ($\partial/\partial t \equiv 0$!) quantum hydrodynamic equations,[7] whereas the steady-state Schrödinger equation with $\partial/\partial t \equiv 0$ appears to have no physical meaning, i.e.,

$$-(\hbar^2/2m)\, \nabla^2\psi + V(\vec{r})\psi = i\hbar\partial\psi/\partial t \neq 0 \tag{46}$$

where $V(\vec{r})$ is the potential energy. According to Einstein,[9] the complex (unreal) time-dependent wave function is nothing else but an ingeneous mathematical means for solving nonlinear steady-state quantum problems. Thus, for so-called quasi-stationary quantum systems (particle

in box, hydrogen atom, harmonic oscillator, etc.)[6] with discrete energy eigenvalues, the Schrödinger equation gives (nonrelativistically) correct results, since the latter systems are true steady-state systems in the hydrodynamic picture of quantum mechanics.[7] For such steady-state systems, the parabolic and hyperbolic Schrödinger equations should lead to the same results, as in classical physics where, e.g., the parabolic and hyperbolic diffusion equations give identical steady-state solutions.

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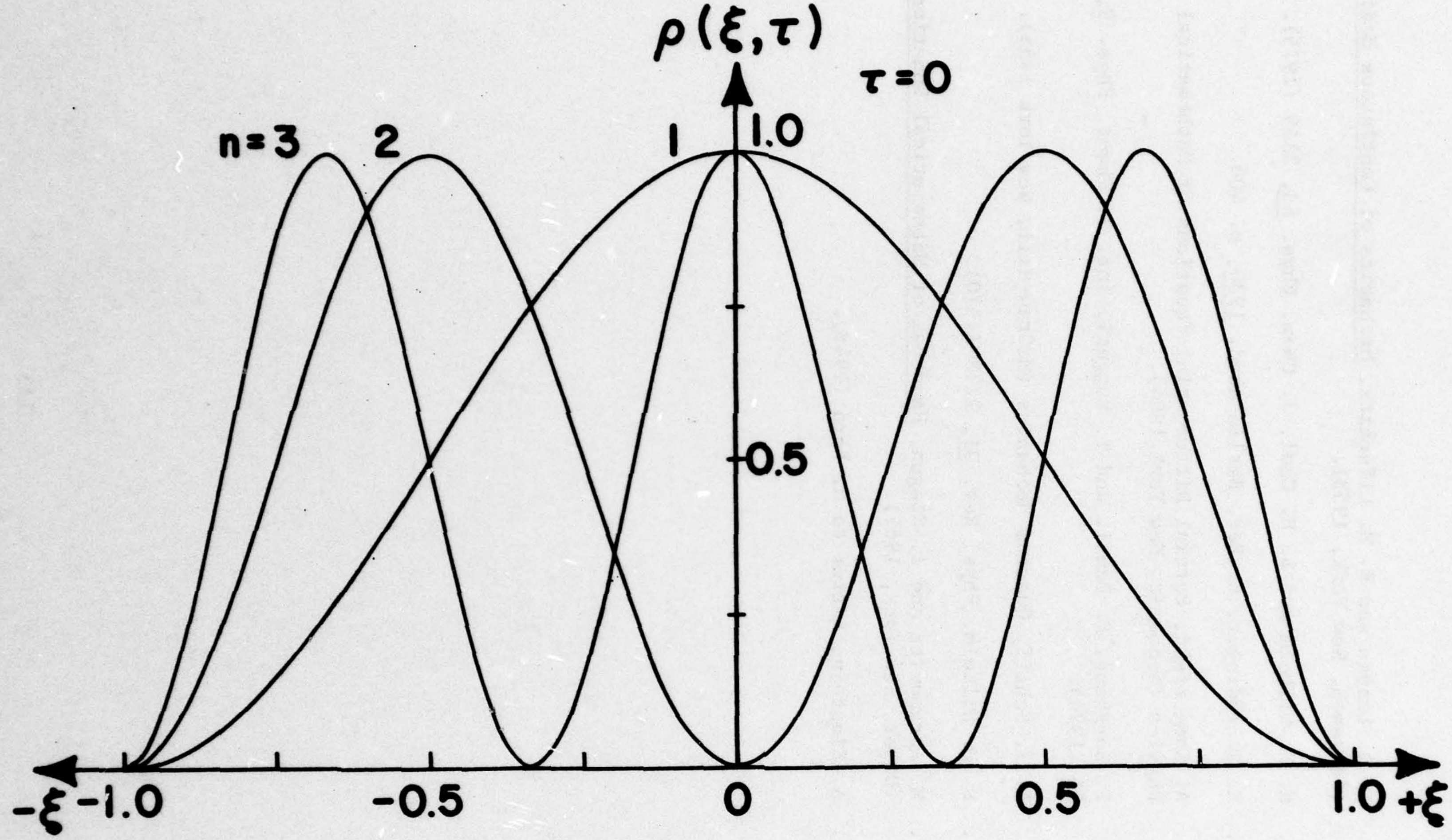

Fig. 1: $\rho(\xi,\tau)$ versus ξ for $\tau = 0$ and $n = 1; 2; 3$.

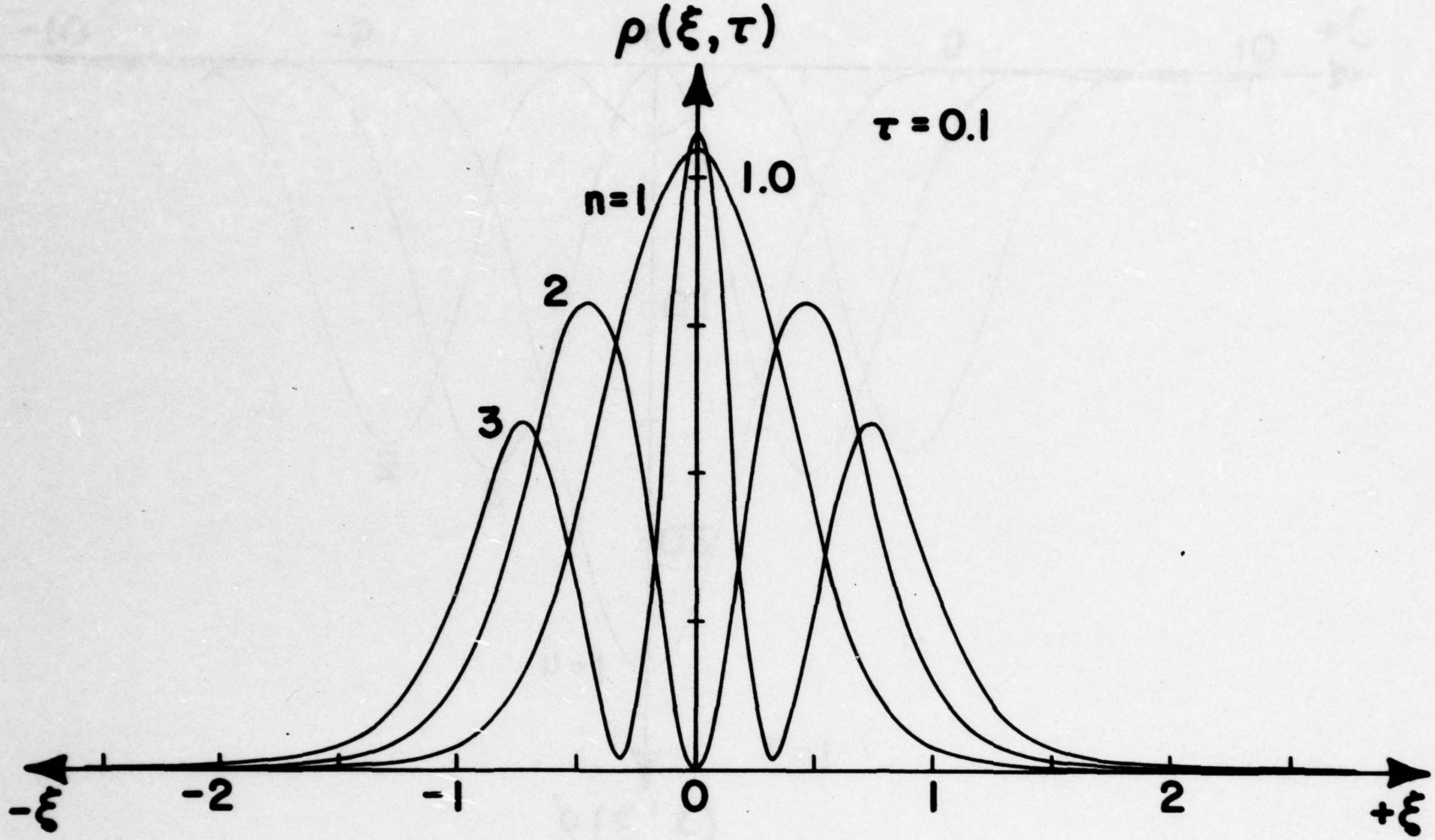

Fig. 2: $\rho(\xi,\tau)$ versus ξ for $\tau = 10^{-1}$ and $n = 1; 2; 3$.

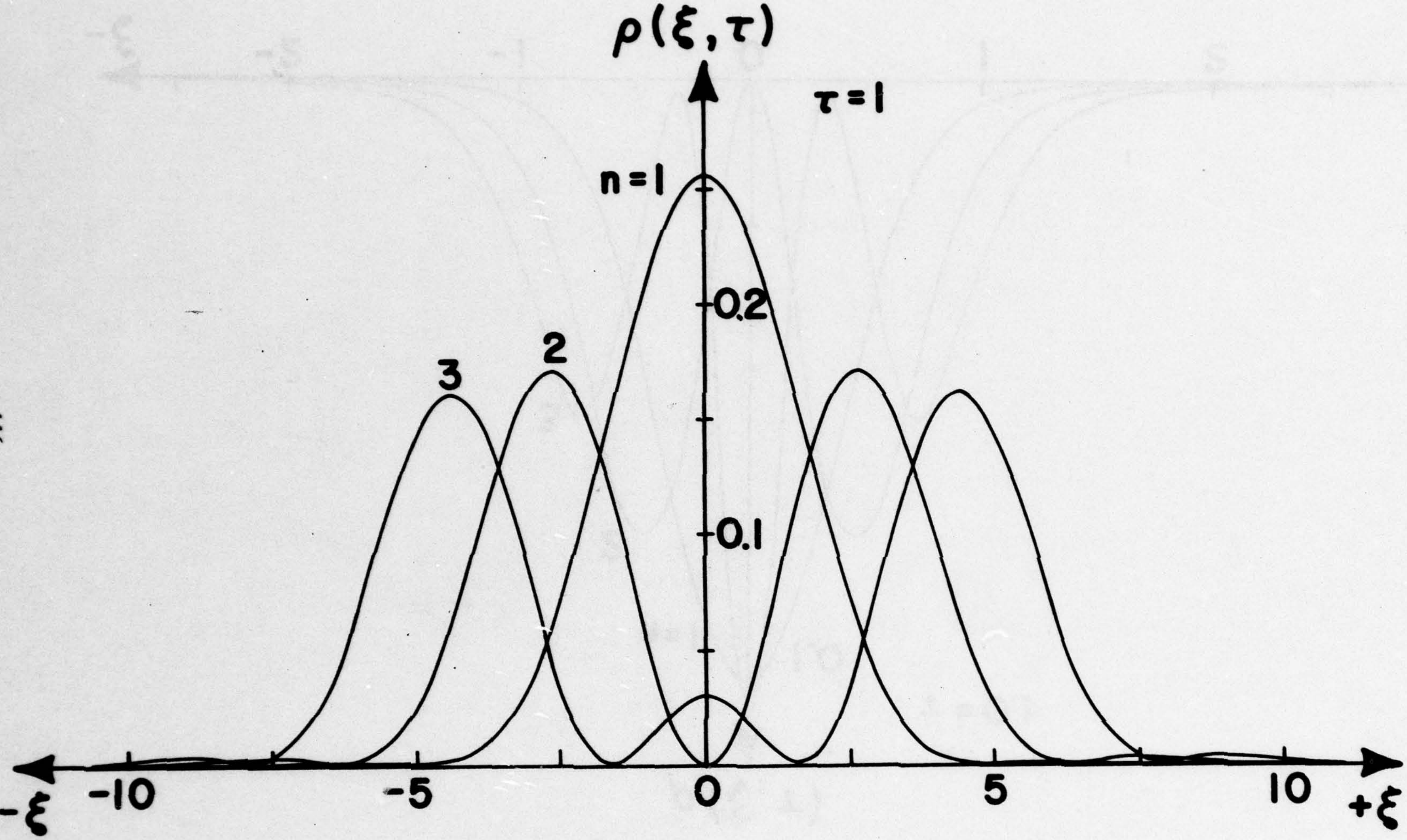

Fig. 3: $\rho(\xi,\tau)$ versus ξ for $\tau = 10^0$ and $n = 1; 2; 3$.

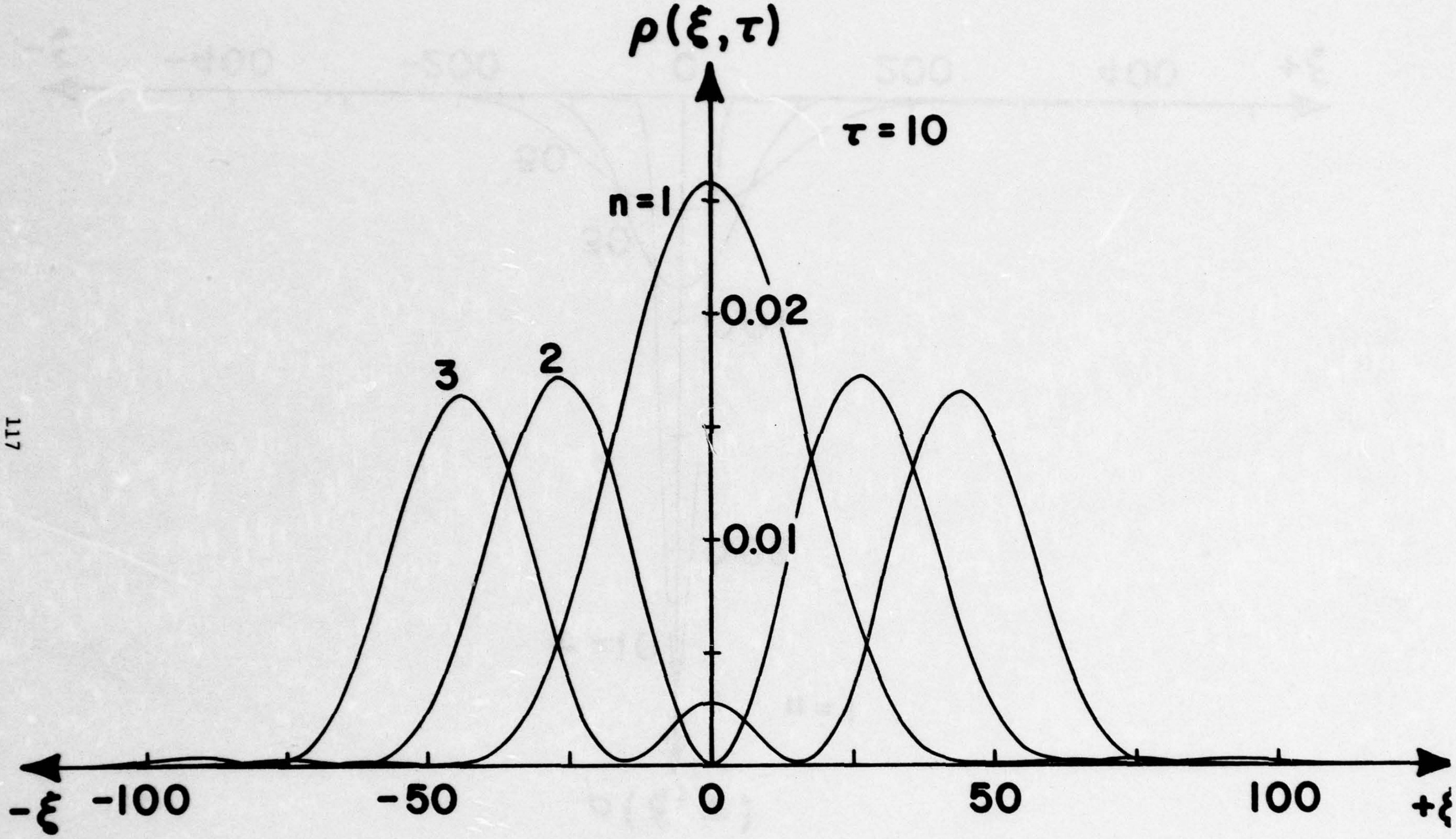

Fig. 4: $\rho(\xi,\tau)$ versus ξ for $\tau = 10^1$ and n = 1; 2; 3.

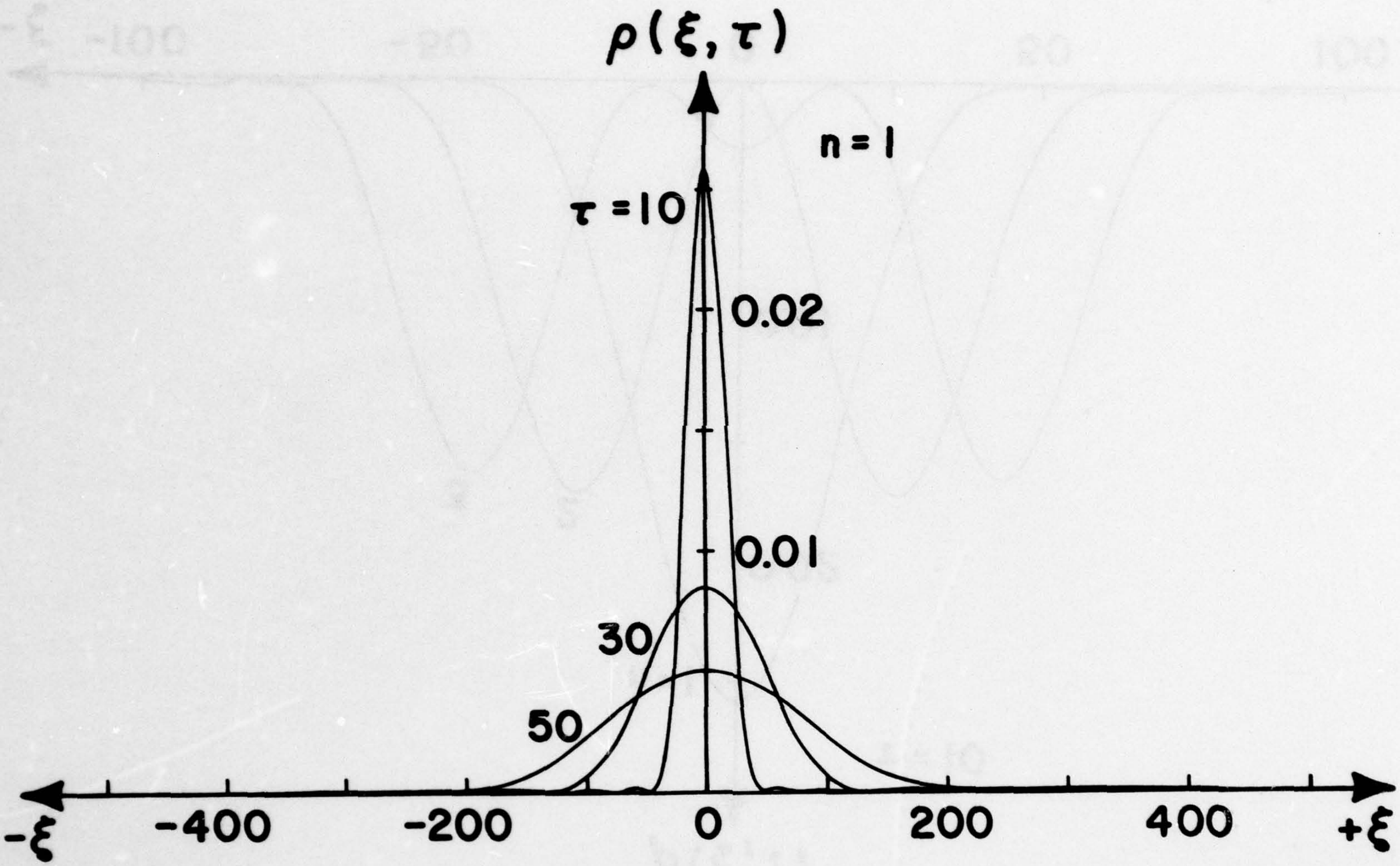

Fig. 5: $\rho(\xi,\tau)$ versus ξ for τ = 10; 30; 50 and n = 1.

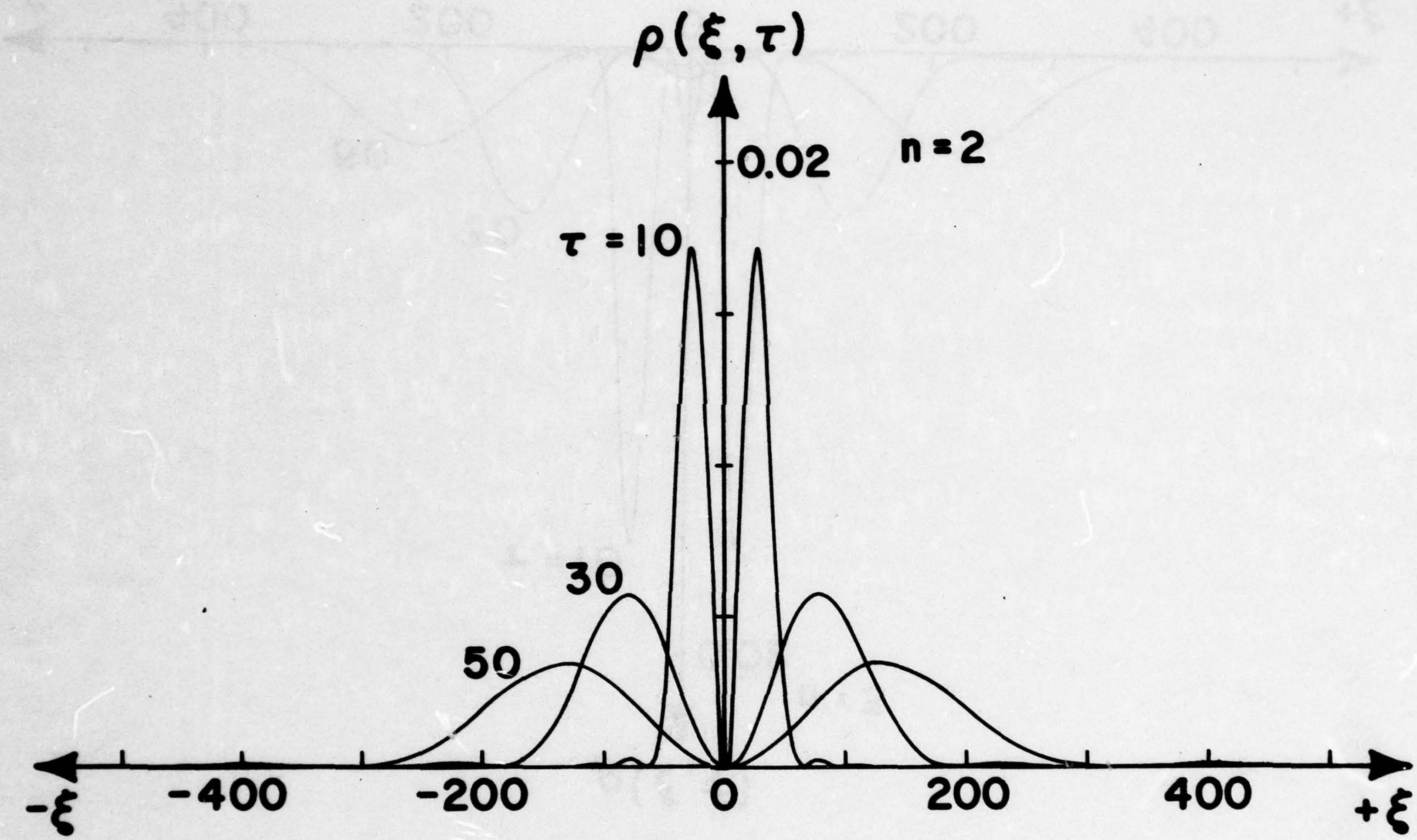

Fig. 6: $\rho(\xi,\tau)$ versus ξ for τ = 10; 30; 50 and n = 2.

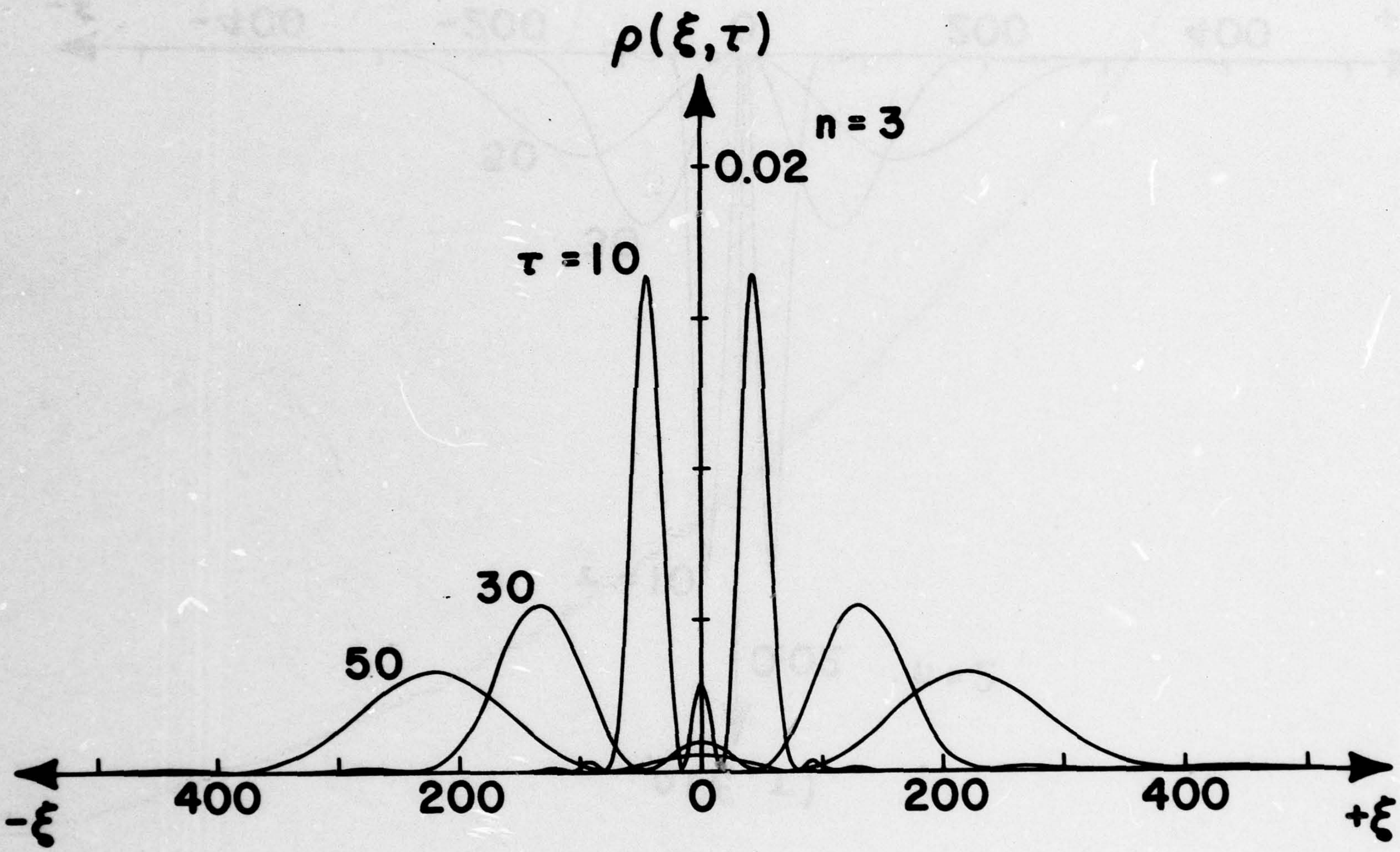

Fig. 7: $\rho(\xi,\tau)$ versus ξ for $\tau = 10; 30; 50$ and $n = 3$.

VII. APPENDIX

INCOMPRESSIBLE MAGNETOHYDRODYNAMIC FLOW WITH HEAT TRANSFER BETWEEN INCLINED WALLS

Abstract

Self-similar flows of viscous, incompressible, electrically conducting fluids and of heat in a diffuser with an azimuthal magnetic field and plane walls (at constant temperature) are analyzed by analytical and numerical methods. Velocity distributions for pure outflows and temperature distributions are presented for various invariance parameters $I_1 = \theta_o R^{1/2}$, $I_2 = (H^2-4)/R$, and Prandtl numbers $\mathcal{P}$ (θ_o = duct angle, R = Reynolds number, H = Hartmann number). In the approximation of constant fluid properties, the onset of flow separation is independent of the heat transfer, and separation is inhibited for $I_2 \geq 2/3$ if $R >> 1$. The theory is applicable to proper incompressible fluids such as liquid metals (small Prandtl numbers), in which self-similar velocity and temperature fields can be realized.

I. INTRODUCTION

When an incompressible, viscous, electrically conducting fluid such as a liquid metal flows through a channel in the presence of a magnetic field, electric currents are induced. As a consequence, the flow field is modified by the Lorentz force, and additional thermal energy is supplied to the conducting fluid through Ohmic heating. In addition, flow energy is dissipated by heat conduction and viscous heating. Accordingly, the temperature distribution in a conducting fluid differs considerably from that in a nonconducting fluid in which flow energy is dissipated only by heat conduction and viscous heating.

In ordinary fluid dynamics, the temperature distribution in an incompressible nonconducting flow between two parallel plates is well understood. [1)] The corresponding magnetohydrodynamic heat transfer problem in the presence of a transverse magnetic field has been treated[2-4)] for constant wall temperature. The nonlinear flow with heat transfer in nonconducting, incompressible fluids between nonparallel walls (Jeffery-Hamel flow) represents one of the few exact solutions of the coupled Navier-Stokes and heat transport equations obtained in the literature.[5)]

The dynamic behavior of the conducting fluid in the magnetohydrodynamic Jeffery-Hamel flow with an azimuthal magnetic field has been studied for isothermal, incompressible[6-8)], and compressible flows in the polytropic approximation[9-10)], i.e., without solving the complete heat transport equation. A numerical analysis of the temperature field in compressible magnetohydrodynamic diffuser flows has been given for variable wall temperature[11)].

In the following, we analyze the heat transfer and fluid dynamics in the magnetohydrodynamic Jeffery-Hamel flow, which deals with

electrically conducting, viscous, incompressible fluids. As usual in incompressible flow theory, the viscosity, thermal and electrical conductivities are assumed to be constant. The temperature at the walls is assumed to be a given constant. The onset of magnetohydrodynamic flow separation is discussed. As a rough approximation, the theory is also applicable to subsonic, quasi-incompressible plasma flows, presumed that the Prandtl number P and interaction parameter I_2 (ratio of magnetic to viscous forces) are sufficiently small.

II. MATHEMATICAL FORMULATION

The electrically conducting fluid is injected through the inner duct section at $r = r_0 > 0$ and removed downstream through the outer duct section at $r = r_1 < \infty$ as shown in Fig. 1. The diffuser has two parallel electrodes in the planes $z = \pm z_\infty$ and two non-parallel, insulating walls in the planes $\theta = \pm\theta_o$. The electrodes are short-circuited so that the electric potential difference between them is zero. The boundary layers at the electrodes are disregarded assuming that the inter-electrode spacing is large, $z_\infty \gg \frac{1}{2}(r_0+r_1)\theta_o$. In this model, all flow fields are two-dimensional. Similarly, hydrodynamic and electromagnetic edge effects at $r = r_0, r_1$ are not considered. The external magnetic field has its source in an electric current I_o flowing through an infinitely long conducting rod along the z axis, and is given by (μ_o is the permeability of vacuum),

$$B_\theta = \frac{\mu_o I_o}{2\pi r} \quad . \tag{1}$$

The induced magnetic field is neglected compared to the external magnetic field, assuming that the magnetic Reynolds number is small, $R_m = \mu_o \sigma u_o r_o \ll 1$, a condition which is satisfied in many cases.

The radial flow across the azimuthal magnetic field induces an electric current in the axial direction, $J_z = \sigma(E_z + uB_\theta)$, if the Hall effect is negligible, $\omega\tau \ll 1$ ($\omega = eB/m$ is the gyration frequency and τ is the collision time of the electrons). The axial electric field E_z vanishes due to the short-circuited external circuit. The resulting Lorentz force is in the radial direction so that the flow is purely radial in the absence of the Hall effect.

According to the magnetohydrodynamic equations[12)], the dimensionless velocity $\bar{u}(\bar{r},\theta)$, pressure $\bar{p}(\bar{r},\theta)$, and temperature $\bar{T}(\bar{r},\theta)$ fields in

the viscous, incompressible, conducting flow across the external magnetic field $B_\theta(r)$ are described by the system of nonlinear, coupled partial differential equations:

$$\frac{1}{\bar{r}}\frac{\partial}{\partial \bar{r}}(\bar{r}\bar{u}) = 0 \quad , \tag{2}$$

$$\bar{u}\frac{\partial \bar{u}}{\partial \bar{r}} = -\frac{\partial \bar{p}}{\partial \bar{r}} + \frac{1}{R}\frac{1}{\bar{r}^2}\frac{\partial^2 \bar{u}}{\partial \theta^2} - \frac{H^2}{R}\frac{\bar{u}}{\bar{r}^2} \quad , \tag{3}$$

$$0 = -\frac{1}{\bar{r}}\frac{\partial \bar{p}}{\partial \theta} + \frac{1}{R}\frac{2}{\bar{r}^2}\frac{\partial \bar{u}}{\partial \theta} \quad , \tag{4}$$

$$P R\bar{u}\frac{\partial \bar{T}}{\partial \bar{r}} = \frac{\partial^2 \bar{T}}{\partial \bar{r}^2} + \frac{1}{\bar{r}}\frac{\partial \bar{T}}{\partial \bar{r}} + \frac{1}{\bar{r}^2}\frac{\partial^2 \bar{T}}{\partial \theta^2} + P H^2 \frac{\bar{u}^2}{\bar{r}^2} + P\left[2\left(\frac{\partial \bar{u}}{\partial \bar{r}}\right)^2 + \left(\frac{1}{\bar{r}}\frac{\partial \bar{u}}{\partial \theta}\right)^2 + 2\left(\frac{\bar{u}}{\bar{r}}\right)^2\right]. \tag{5}$$

The Reynolds (R), Hartmann (H), and Prandtl (P) numbers, and the dimensionless variables are defined by

$$R = \frac{\rho_o u_o r_o}{\mu} \; , \; H^2 = \frac{\sigma}{\mu}\left(\frac{\mu_o I_o}{2\pi}\right)^2 \; , \; P = \frac{\mu C_v}{\kappa} \quad , \tag{6}$$

and

$$\bar{u} = u/u_o \; , \; \bar{p} = p/\rho_o u_o^2 \; , \; \bar{T} = (T-T_w)/(u_o^2/C_v), \; \bar{r} = r/r_o \quad . \tag{7}$$

ρ_o is the mass density, μ the viscosity, σ the electrical conductivity, C_v the specific heat capacity at constant volume, κ the thermal conductivity, and T_w is the wall temperature. u_o and r_o are reference values for the velocity and radial distance, respectively.

The flow fields in Eqs. (2)-(5) are subject to the boundary and symmetry conditions:

$$\bar{u}(\bar{r}, \pm\theta_o) = 0 \quad , \tag{8}$$

$$\bar{T}(\bar{r}, \pm\theta_o) = 0 \quad , \tag{9}$$

$$\bar{u}(\bar{r}, +\theta) = \bar{u}(\bar{r}, -\theta) \quad , \tag{10}$$

$$\bar{T}(\bar{r},+\theta) = \bar{T}(\bar{r},-\theta) \quad . \tag{11}$$

Equation (8) is the no-slip condition for the radial velocity at the walls. Equation (9) considers that the temperature of the fluid at the walls equals the wall temperature. Equations (10) and (11) are the symmetry conditions for the velocity and temperature field, respectively. Although the Jeffery-Hamel flow is known to have also non-symmetrical solutions[5)], we consider herein exclusively symmetrical flows. The flow rate per unit height (Δz=1) in dimensionless form is positive for outflows,

$$\bar{Q} = R \int_{-\theta_o}^{+\theta_o} \bar{u}\bar{r}d\theta > 0 \quad . \tag{12}$$

Self-similar solutions for the velocity field are physically meaningful in many cases because end effects decay rapidly with distance from the ends so that self-similar solutions are realized within the main flow region. The same is true for the temperature field if thermal end effects decay rapidly with distance from the ends, which requires large thermal conduction, i.e. sufficiently small Prandtl numbers P.

III. SELF-SIMILAR SOLUTION FOR $\bar{u}(\bar{r},\theta)$

Integration of Eq. (4) with respect to θ, and elimination of the pressure field from Eq. (3) yield

$$\bar{p} = \frac{2}{R}\frac{\bar{u}}{\bar{r}} + \beta(\bar{r}) \quad , \tag{13}$$

and

$$\bar{u}\frac{\partial\bar{u}}{\partial\bar{r}} = \frac{1}{R}\left(\frac{1}{\bar{r}^2}\frac{\partial^2\bar{u}}{\partial\theta^2} - \frac{2}{\bar{r}}\frac{\partial\bar{u}}{\partial\bar{r}} + 2\frac{\bar{u}}{\bar{r}^2}\right) - \frac{H^2}{R}\frac{\bar{u}}{\bar{r}^2} - \frac{d\beta(\bar{r})}{d\bar{r}} \quad . \tag{14}$$

The continuity equation (2) indicates that the radial velocity field is of the self-similar form

$$\bar{u}(\bar{r},\theta) = f(\theta)/\bar{r} \quad . \tag{15}$$

Equation (15) reduces Eq. (14) to the ordinary differential equation

$$\frac{d^2f}{d\theta^2} + (4-H^2)\,f + Rf^2 = R\bar{r}^3\frac{d\beta}{d\bar{r}} \equiv -\alpha R \tag{16}$$

where α is a separation constant. Upon introducing the independent variable $x = \theta/\theta_o$, Eq. (16) gives

$$\frac{d^2f}{dx^2} + \theta_o^2(4-H^2)f + R\theta_o^2f^2 + \alpha R\theta_o^2 = 0 \quad , \tag{17}$$

and

$$\beta(\bar{r}) = \frac{1}{2}\frac{\alpha}{\bar{r}^2} + p_\infty \quad , \tag{18}$$

where

$$f(\pm 1) = 0 \quad , \tag{19}$$

$$\left.\frac{df}{dx}\right|_{x=0} = 0 \quad , \tag{20}$$

by Eqs. (8) and (10). p_∞ is an integration constant representing the hydrostatic overpressure. The flow rate in Eq. (12) is rewritten as

$$\bar{Q}/\theta_o = R \int_{-1}^{+1} f(x)dx \quad . \tag{21}$$

If the flow rate is given, Eq. (21) would have to be used as a constraint for f(x).

However, for computations it is more convenient to assume the Reynolds number $R \equiv R(0) = \rho_o u(r,\theta=0)r/\mu$ of the central stream line to be given[5)] which implies the simpler condition

$$f(0) = 1 \quad . \tag{22}$$

We seek symmetrical solutions to the boundary-value problem in Eqs. (17), (19), (20) and (22) which represent pure outflows, $0 \leq f(x) \leq 1$. A first integration of Eq. (17) gives

$$\left(\frac{df}{dx}\right)^2 = \frac{2}{3} R\theta_o^2 \left[C - f^3 - \frac{3}{2}R^{-1}(4-H^2)f^2 - 3\alpha f\right] \tag{23}$$

where $[df(\pm1)/dx]^2 = (2/3)R\theta_o^2 C \geq 0$ for $R > 0$, since $f(\pm1) = 0$. Since $df(0)/dx = 0$ and $f(0) = 1$, Eq. (23) gives

$$C = 1 + \frac{3}{2}R^{-1}(4-H^2) + 3\,\alpha \geq 0 \quad . \tag{24}$$

Substitution of Eq. (24) and integration of Eq. (23) yields

$$\left(\frac{2}{3}\right)^{1/2} I_1 x = \pm \int_1^{f(x)} \frac{df}{\sqrt{-G(f)}} \quad , \tag{25}$$

where

$$G(f) = (f^3-1) - \frac{3}{2}\,I_2(f^2-1) + 3\alpha(f-1) \quad , \tag{26}$$

$$I_1 = \sqrt{R}\,\theta_o \quad , \quad I_2 = R^{-1}(H^2-4) \quad . \tag{27}$$

Note that $I_2 \doteq \frac{H^2}{R}$ for $H^2 \gg 4$, i.e., I_2 is essentially the magnetohydrodynamic interaction parameter, since H^2 has the same magnitude as $R \gg 1$ in flows with significant magnetic body forces as of practical interest. According to Eqs. (24) and (27), the eigenvalues α of pure

outflows are larger than a critical value $\tilde{\alpha}$,

$$\alpha \geq \tilde{\alpha} \ , \quad \tilde{\alpha} = -\frac{1}{3}(1-\frac{3}{2}I_2) \ , \tag{28}$$

with

$$\tilde{\alpha} \gtrless 0 \text{ for } I_2 \gtrless \frac{2}{3} \ . \tag{29}$$

The trinomial in Eq. (26) has the *roots*,

$$f = 1 \ , \quad f = f_+ \ , \quad f = f_- \tag{30}$$

where

$$f_\pm = \frac{1}{2}\{-(1-\frac{3}{2}I_2) \pm [(1-\frac{3}{2}I_2)^2 - 12\alpha - 4(1-\frac{3}{2}I_2)]^{\frac{1}{2}}\} \ , \tag{31}$$

with

$$f_\pm = \begin{matrix}\text{complex conjugate} \\ \text{real}\end{matrix}\} \text{ for } \alpha \begin{matrix}> \\ \leq\end{matrix} -\frac{1}{4}(1-\frac{3}{2}I_2)(1+\frac{1}{2}I_2) \ . \tag{32}$$

It is noted that the product of the three roots is always positive, $1 \cdot f_+ \cdot f_- \geq 0$. In view of Eq. (30), Eq. (25) is rewritten as

$$(2/3)^{\frac{1}{2}} I_1 x = \pm \int_1^{f(x)} [(-1)(f-f_1)(f-f_2)(f-f_3)]^{-\frac{1}{2}} df \ . \tag{33}$$

In the evaluation of the elliptic integral [13] in Eq. (33) for pure outflows, $0 \leq f(x) \leq 1$, two cases are distinguished:

Case 1: $f_\pm$ = complex conjugate

In this case, the eigenvalue α lies in the region

$$-(2-3I_2)(2+I_2)/6 < \alpha < \infty \tag{34}$$

by Eq. (32). The substitution $f = 1 - \lambda^2[(1-\cos\phi)/(1+\cos\phi)]$, $0 \leq \phi \leq \pi$, reduces Eq. (33) to $(2/3)^{\frac{1}{2}} I_1 x = \mp\lambda^{-1} F(\phi,k)$. The solution $f(x)$ is obtained by inversion of the elliptic integral $F(\phi,k)$ of the first kind as:

$$f(x) = 1 - \lambda^2 \frac{1-\mathrm{cn}[(2/3)^{\frac{1}{2}} I_1 \lambda x;k]}{1+\mathrm{cn}[(2/3)^{\frac{1}{2}} I_1 \lambda x;k]} \quad , \tag{35}$$

where

$$\lambda^2 = \{3[1+\alpha-I_2]\}^{\frac{1}{2}} \quad , \tag{36}$$

$$k^2 = \frac{1}{2}[1 + \frac{3}{4}\lambda^{-2}(2-I_2)] \quad . \tag{37}$$

The eigenvalue α is determined by the boundary condition (19) as the real root of the transcendental equation,

$$\mathrm{cn}[(2/3)^{\frac{1}{2}} I_1 \lambda;k] = (\lambda^2-1)/(\lambda^2+1) \quad ,$$

or

$$(2/3)^{\frac{1}{2}} \lambda I_1 = \mp F(\phi,k), \quad \phi = \mathrm{arc\ cos}[(\lambda^2-1)/(\lambda^2+1)] \quad . \tag{38}$$

Case 2: $f_{\pm}$ = real

In this case, $f_+ > f_-$ by Eq. (31), and $f_{\pm} \geq 0$ for $I_2 \geq 2/3$ and $f_{\pm} \leq 0$ for $I_2 \leq 2/3$. In order to evaluate the elliptic integral in Eq. (33), the ordering of the roots is chosen such that $f_1 > f_2 > f_3$.

(a) $f_{\pm} \geq 0$: If $f_3 = 1$, then $f_1 = f_+$ and $f_2 = f_-$. Since $f_2 > f_3 = 1$, $\alpha \geq \tilde{\alpha}$ [Eq. (28)], and $I_2 \geq 2/3$ [Eq. (29)], the eigenvalue α lies in the interval

$$\max[(I_2-1);\ (3I_2-2)/6] \leq \alpha \leq (3I_2-2)(I_2+2)/16;\ I_2 \geq 2/3 \quad . \tag{39}$$

The substitution $f = f_2 - (f_2-1)/\cos^2\phi$, $0 \leq \phi \leq \pi/2$, reduces Eq. (33) to $(2/3)^{\frac{1}{2}} I_1 x = \mp\lambda^{-1} F(\phi,k)$. Inversion of the elliptic integral yields the solution:

$$f(x) = f_2-(f_2-1)\mathrm{cn}^{-2}[(2/3)^{\frac{1}{2}} \lambda I_1 x;k] \quad , \tag{40}$$

where

$$\lambda^2 = (f_1-1)/4 \quad , \tag{41}$$

$$k^2 = (f_1-f_2)/(f_1-1) \quad . \tag{42}$$

The eigenvalue α is determined by the boundary condition (19) as the real root of the transcendental equation,

$$cn^2[(2/3)^{\frac{1}{2}}\lambda I_1;k] = (f_2-1)/f_2 \quad ,$$

or

$$(2/3)^{\frac{1}{2}}\lambda I_1 = \mp F(\phi,k), \quad \phi = \text{arc}\cos\{[(f_2-1)/f_2]^{\frac{1}{2}}\} \quad . \tag{43}$$

<u>(b) $f_\pm \leq 0$:</u> Since $0 \geq f_+ > f_-$, it is $f_1 = 1$, $f_2 = f_+$, and $f_3 = f_-$. By Eqs. (28) and (32), the eigenvalues α lies in the interval

$$(3I_2-2)/6 \leq \alpha \leq (3I_2-2)(I_2+2)/16; \; I_2 \leq \frac{2}{3} \quad . \tag{44}$$

The substitution $f = f_2 + (1-f_2)\cos^2\phi$, $0 \leq \phi \leq \pi/2$, reduces Eq. (33) to $(2/3)^{\frac{1}{2}}I_1 x = \mp\lambda^{-1}F(\phi,k)$. Inversion of the elliptic integral yields the solution:

$$f(x) = f_2 + (1-f_2)cn^2[(2/3)^{\frac{1}{2}}\lambda I_1 x;k] \quad , \tag{45}$$

where

$$\lambda^2 = (1-f_3)/4 \quad , \tag{46}$$

$$k^2 = (1-f_2)/(1-f_3) \quad . \tag{47}$$

The eigenvalue α is determined by the boundary condition (19) as the real root of the transcendental equation,

$$cn^2[(2/3)^{\frac{1}{2}}I_1\lambda;k] = -f_2/(1-f_2) \quad ,$$

or

$$(2/3)^{\frac{1}{2}}I_1\lambda = \mp F(\phi,k), \quad \phi = \text{arc}\cos\{[f_2/(1-f_2)]^{\frac{1}{2}}\} \quad . \tag{48}$$

The Eqs. (34), (39), and (44) show that solutions f(x) for the entire spectrum of eigenvalues, $\tilde{\alpha} \leq \alpha < \infty$, have been obtained.

In the limiting case of vanishing magnetic field or electrical conductivity, i.e., $H^2 \to 0$ and $I_2 \to -4/R$, the solutions in Eq. (35) for $f_\pm$ = complex conjugate and in Eq. (45) for $f_\pm$ = real ≤ 0 are reduced to those of the Jeffery-Hamel flow for nonconducting fluids[5] or for

conducting fluids in a homogeneous axial magnetic field[14].

The self-similar solutions derived under cases (1) and (2) in Eqs. (34)-(38), (39)-(43), and (44)-(48) exhibit a fundamental invariance principle. If the flow parameters R, H and θ_o are varied such that the values of the combinations $I_1 = R^{1/2}\theta_o$ and $I_2 = (H^2-4)/R$ are unchanged, then λ, k, α, and, hence, the solution f(x) remain unchanged, too. For large Reynolds numbers, R >> 1, the invariance parameter I_2 becomes the magnetohydrodynamic interaction parameter defined by $I = \sigma B_o^2 r_o/\rho_o u_o = H^2/R$. This invariance principle has been first found for compressible magnetohydrodynamic flows[9].

In Fig. 2, the eigenvalue α for pure outflows is plotted versus the invariance parameter $I_1 = R^{1/2}\theta_o$ for various values of the invariance parameter $I_2 = H^2/R$ *for* R >> 1, *where* the dotted line gives α for $H^2 = 0$ [based on Eqs. (38), (43), and (48)]. For fixed I_2, α decreases with increasing I_1. For fixed I_1, α is the larger the larger I_2 is.

Figs. 3 and 4 show the velocity distributions f(x) versus x for pure outflows with I_1 and I_2 (I_2 = 0.1 and 0.5, respectively) as parameters [based on Eqs. (35) and (45)]. In these cases, the magnetic infraction is small, and f(x) has a parabolic shape. It is seen how f(x) changes from a parabolic profile for small values of $I_1 = R^{1/2}\theta_o$ to the critical flow with maximum value of I_1, at which the transition to mixed flows occurs.

Fig. 5 shows f(x) versus x with various values of I_1, for the critical value I_2 = 2/3 for R >> 1, above which flow separation is inhibited[6-9].

Figs. 6 and 7 give f(x) versus x with I_1 as a parameter for large values of I_2(I_2=3, and 5, respectively). In these cases, the

magnetohydrodynamic interaction is dominant so that flow separation does not occur. The velocity profile f(x) changes from a parabolic shape to a rectangular one as I_1 increases.

IV. FLOW SEPARATION

At the transition from pure outflow, $f(x) > 0$, to backflow $f(x) < 0$ at the walls, the a imuthal velocity gradient drops to zero for $\theta = \pm\theta_o$. This transition point is usually defined as the onset of flow separation in incompressible fluids[1].

Applying the condition $df(\pm 1)/dx = 0$ to the solution in Eq. (45), which is valid for eigenvalues down to the critical value $\tilde{\alpha} = -(2-3I_2)/6 = -(1+6/R - 3H^2/2R)$, yields for the critical value $\tilde{I}_1$ for onset of flow separation,

$$(2/3)^{\frac{1}{2}}\, \tilde{\lambda}\, \tilde{I}_1 = K(\tilde{k}) \tag{49}$$

where $K(\tilde{k})$ is the complete elliptic integral of the first kind[13],

and

$$\tilde{\lambda}^2 = (1-\tilde{f}_3)/4 \quad , \quad \tilde{k}^2 = (1-\tilde{f}_2)/(1-\tilde{f}_3) \quad , \tag{50}$$

$$f_1 = 1 \quad , \quad \tilde{f}_2 = 0 \quad , \quad \tilde{f}_3 = -(1+6/R-3H^2/2R) \quad . \tag{51}$$

The critical value $\tilde{I}_1$ in Eq. (49) determines the critical duct angle $\tilde{\theta}_o$ for a given Reynolds number or the critical Reynolds number $\tilde{R}$ for a given duct angle. Accordingly, the critical duct angle above which $(\theta_o > \tilde{\theta}_o)$ flow separation sets in is given by

$$\tilde{\theta}_o = \left[\frac{6}{(3/2)(4-H^2)+2R}\right]^{\frac{1}{2}} K(\tilde{k}) \quad , \quad \tilde{k} = \left[\frac{R}{(3/2)(4-H^2)+2R}\right]^{\frac{1}{2}} \quad . \tag{52}$$

Equation (52) diverges for $I_2 = (H^2-4)/R \geq \frac{2}{3}$ since $\tilde{k} \geq 1$ for $I_2 \geq \frac{2}{3}$. It is, therefore, recognized that pure outflows exist for the following duct angles

$$\begin{aligned} &0 \leq \theta_o \leq \theta_o(\alpha=\tilde{\alpha}) = \pi \quad , \quad \text{for } I_2 \geq \tilde{I}_2 \quad , \\ &0 \leq \theta_o \leq \theta_o(\alpha=\tilde{\alpha}) < \pi \quad , \quad \text{for } I_2 < \tilde{I}_2 \quad , \end{aligned} \tag{53}$$

where the critical value of I_2 determining the critical Hartmann number

is defined by

$$\tilde{I}_2 = (\tilde{H}^2-4)/R = \frac{2}{3}\ , \text{ or } \tilde{I}_2 = \tilde{H}^2/R = \frac{2}{3} \text{ for } R >> 1 \quad . \tag{54}$$

Fig. 8 shows the critical duct angle $\tilde{\theta}_o$ versus Reynolds number for various values of Hartmann number H. It is seen that the critical duct angle decreases as the Reynolds number R increases. For fixed R, $\tilde{\theta}_o$ is the larger the larger H is.

V. SOLUTION FOR $\bar{T}(\bar{r},\theta)$

Since the transport coefficients μ, σ, and κ are treated as independent of temperature, the energy and momentum equations are uncoupled[5]. Therefore, the temperature distribution $\bar{T}(\bar{r},\theta)$ in the diffuser is found by solving the dimensionless energy conservation equation (5) for given velocity fields $\bar{u}(\bar{r},\theta)$.

According to Eq. (7), the dimensionless variable $\bar{T}(\bar{r},\theta)$ is the temperature difference between the conducting fluid and the walls. The solution to Eq. (5) is sought in the self-similar form,[5]

$$\bar{T}(\bar{r},\theta) = g(x)/\bar{r}^2 \quad , \quad x = \theta/\theta_o \quad . \tag{55}$$

Substituting Eqs. (15) and (55) into Eq. (5) yields a second order ordinary differential equation for g(x) which is linear

$$\frac{d^2g}{dx^2} + R\theta_o^2(\frac{4}{R} + 2\mathcal{P}\, f)g + \mathcal{P}(\frac{df}{dx})^2 + \mathcal{P}\, R\theta_o^2(\frac{4}{R} + \frac{H^2}{R})f^2 = 0 \quad , \tag{56}$$

with

$$g(\pm 1) = 0 \quad , \quad dg(0)/dx = 0 \quad , \tag{57}$$

as boundary and symmetry conditions [Eqs. (9) and (11)]. $f = f(x)$ is the (known) solution for the radial velocity field. The Reynolds (R), Hartmann (H), and Prandtl ($\mathcal{P}$) numbers are defined by Eq. (6). In magnetohydrodynamic heat transfer problems, H^2 represents the ratio of the heat due to ohmic heating to that due to viscous dissipation[12]. It is recognized that the invariance principle also holds for magnetohydrodynamic heat transfer problems when R >> 1. For large Reynolds numbers R >> 1, Eq. (56) reduces to

$$\frac{d^2g}{dx^2} + \mathcal{P}\,[2I_1^2fg + (\frac{df}{dx})^2 + I_1^2I_2f^2] = 0 \quad , \tag{58}$$

where $I_1 = R^{\frac{1}{2}}\theta_o$, and $I_2 = H^2/R$ for $R >> 1$, as before.

The homogeneous part of the differential equation (57) has the form of the generalized Lamé equation[5], $\frac{d^2y}{dx^2} = [n(n+1)A(x)+B]y$, n = integer. If an analytical solution to the homogeneous equation can be found, an analytical solution to the inhomogeneous equation is obtained by the method of the variation of parameters or the method of Green's functions. However, the result is usually of an integral form and is inconvenient for numerical calculations. For this reason, a completely numerical method is used to integrate Eq. (57). Substituting Eqs. (23) and (24) into Eq. (57) gives for $R >> 1$,

$$\frac{d^2g}{dx^2} + P\, I_1^2[2fg + A(x)] = 0 \qquad , \qquad (59)$$

where

$$A(x) \equiv \frac{2}{3}[(1-f^3) - \frac{3}{2}I_2(1-f^2) + 3\alpha(1-f)] + I_2f^2 \qquad . \qquad (60)$$

The integration of Eq. (59) with the boundary conditions in Eq. (57) is carried out by means of the method of central finite differences[16]. Eq. (59) and the boundary conditions are written in the following finite difference equation form neglecting higher order terms[16],

$$g_{j+1} + 2(\Delta^2 P I_1^2 f_j - 1)g_j + g_{j-1} = -\Delta^2 P I_1^2 A_j \qquad (61)$$

with

$$g_{j+1} = g_{j-1} \quad \text{for} \quad x = 0 \qquad , \qquad (62)$$

$$g_j = 0 \quad \text{for} \quad x = \pm 1 \qquad , \qquad (63)$$

where

$$A_j = \frac{2}{3}[(1-f_j^3) - \frac{3}{2}I_2(1-f_j^2) + 3\alpha(1-f_j(] + \frac{I_2}{2}f_j^2 \quad , \qquad (64)$$

and Δ is the interval width.

Numerical solutions to Eqs. (61)-(64) giving the relative temperature amplitude $g(x)$ versus x are shown in Figs. 9 and 10. Fig. 9 shows that $g(x)$ is positive ($T > T_w$) and of parabolic form for sufficiently small $\mathcal{P} = 0.5$, and increases with increasing I_2, i.e. ratio of Ohmic to viscous heating. Fig. 10 for intermediate $\mathcal{P} = 1.0$ shows how the temperature distributions $g(x)$ increase with increasing I_2. As I_2 approaches 5, a temperature field develops which is somewhat smaller than the wall temperature ($g(x) < 0$) in the regions close to the walls $x = \pm 1$. It should be noted that $g(x) < 0$ does not imply that the temperature T itself is negative since in general for incompressible flows

$$T(\bar{r},x) = T_w + (u_o^2/c_v)\,\frac{g(x)}{\bar{r}^2} > 0 \quad .$$

In Figs. 11 and 12, the development of relative temperature distributions $g(x)$ is presented which obviously have no physical meaning. Fig. 11 for $\mathcal{P} = 2$ shows the occurrence of hypothetical temperature distributions $g(x) > 0$ with maxima and a minimum as I_2 increases from $I_2 = 1$ to 5. Since $\mathcal{P} = 2$ would imply a plasma, the assumed quasi-incompressibility for subsonic flow speeds does no longer exist due to the large temperature changes for $I_2 > 1$. Fig. 12 for small $I_2 = 0.1$ shows the occurrence of hypothetical temperature distributions $g(x) \gtrless 0$ with one maximum and two minima as $\mathcal{P}$ increases from 0.1 to 1.5. $\mathcal{P} > 0.1$ would require a plasma as the working fluid and the calculated large temperature fluctuations $g(x) \gtrless 0$ are incompatible with the assumed

quasi-incompressibility for subsonic flow speeds. Thus, it is seen that the incompressible theory cannot be applied to subsonic plasma flows, it be P and I_2 are sufficiently small.

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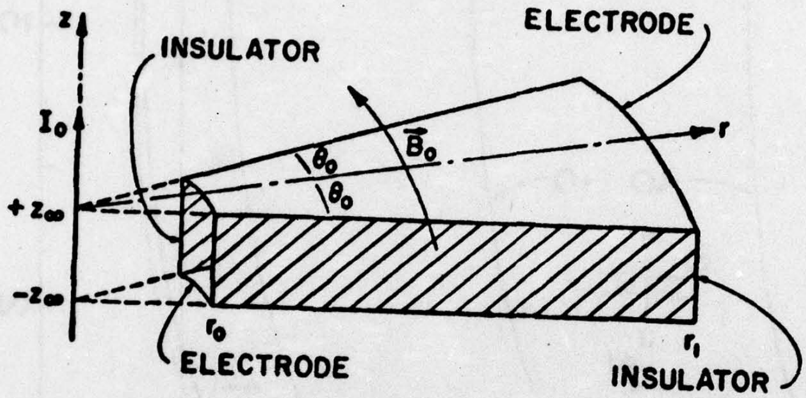

Fig. 1: Diffuser duct geometry.

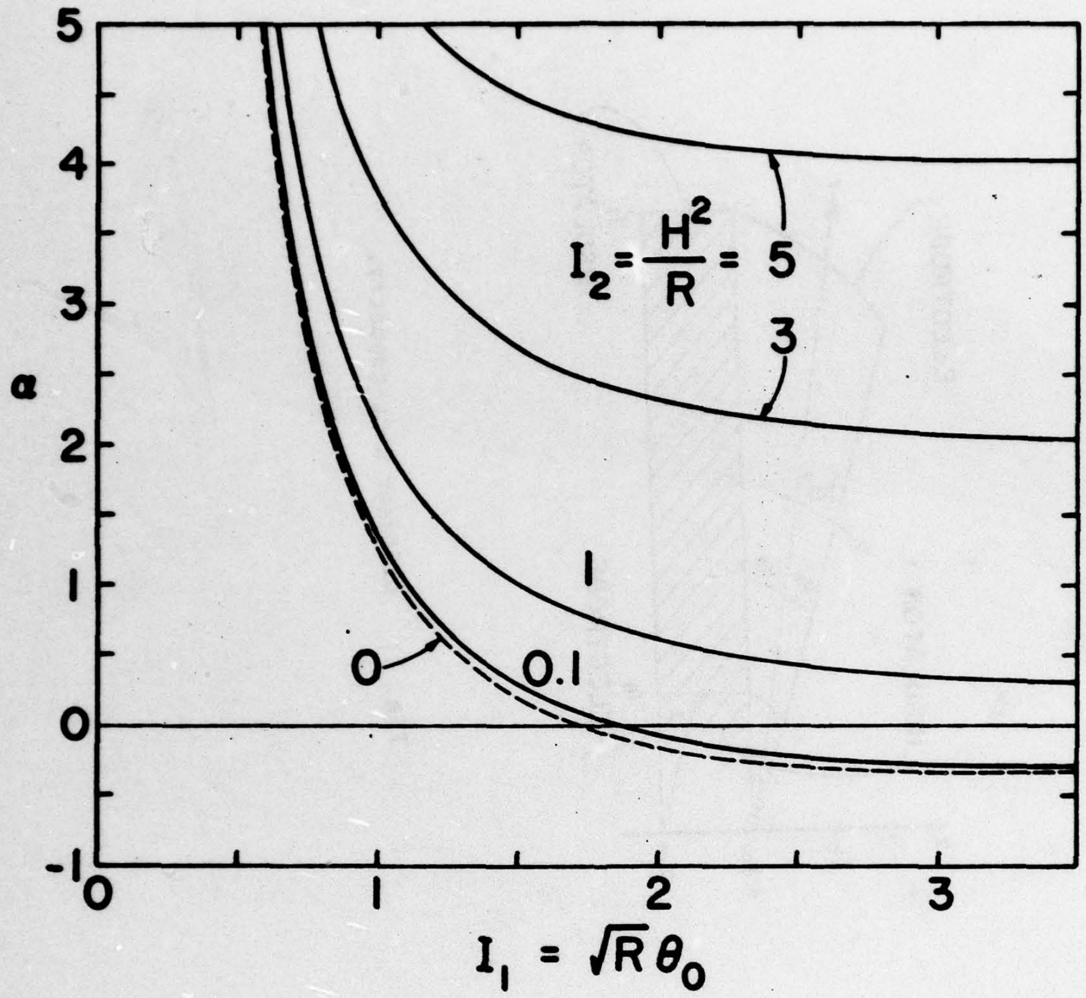

Fig. 2. Eigenvalue α versus invariance parameter I_1 for various interaction parameters I_2.

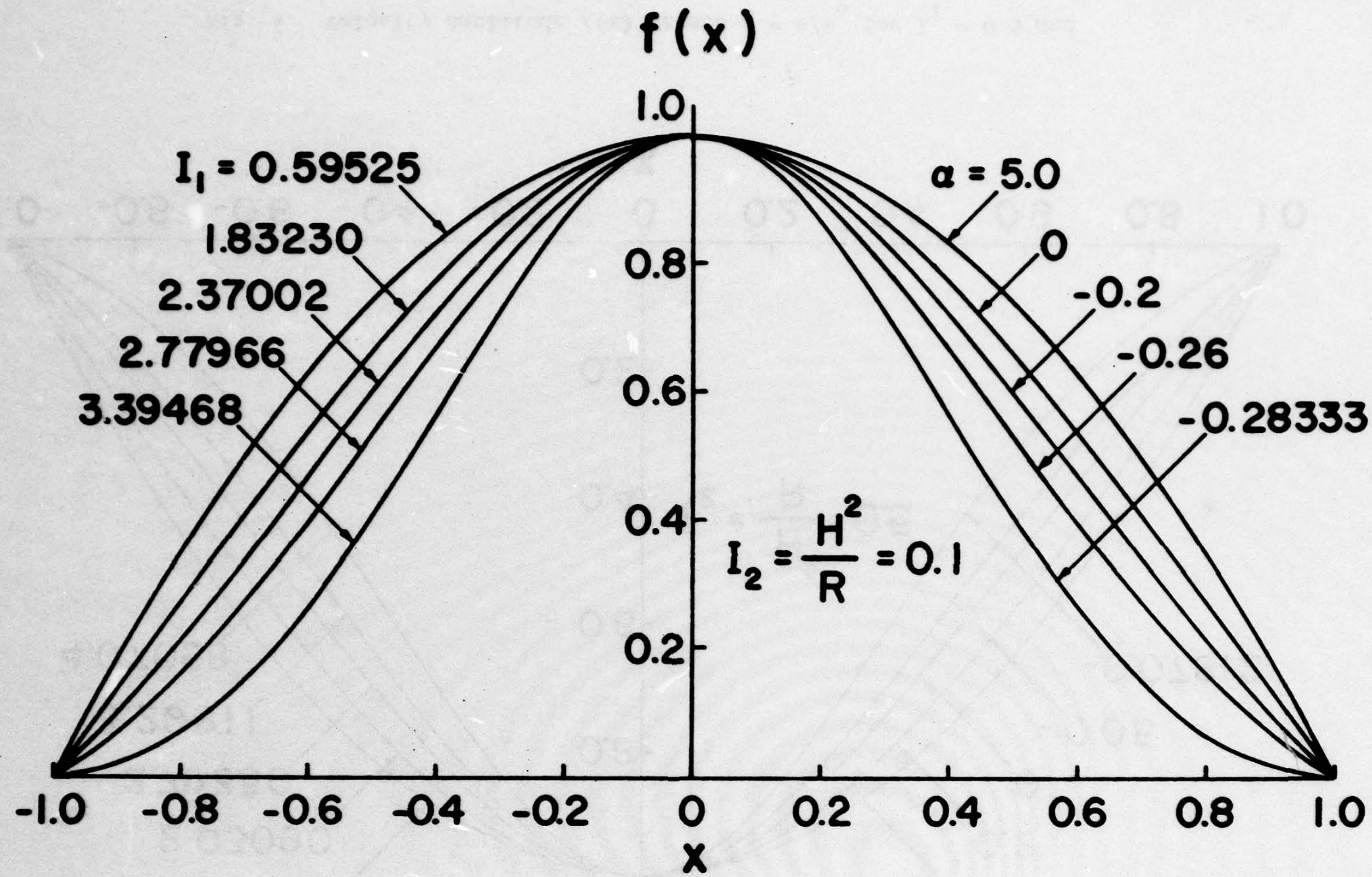

Fig. 3. Velocity amplitude $f(x)$ versus $x = \theta/\theta_o$ for $I_2 = 0.1$ and various $I_1 = R^{1/2}\theta_o$.

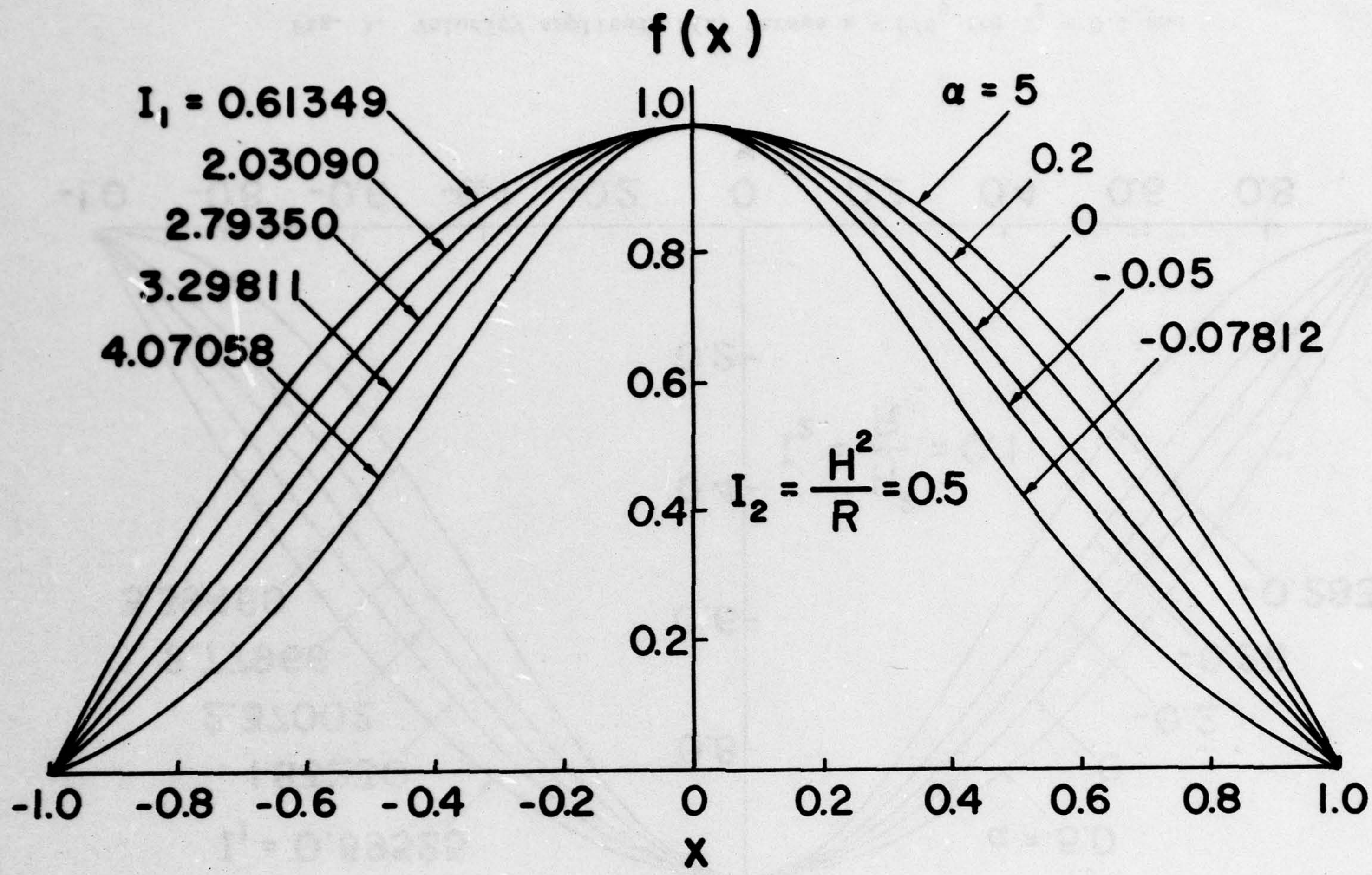

Fig. 4. Velocity amplitude f(x) versus $x = \theta/\theta_o$ for $I_2 = 0.5$ and various $I_1 = R^{1/2}\theta_o$.

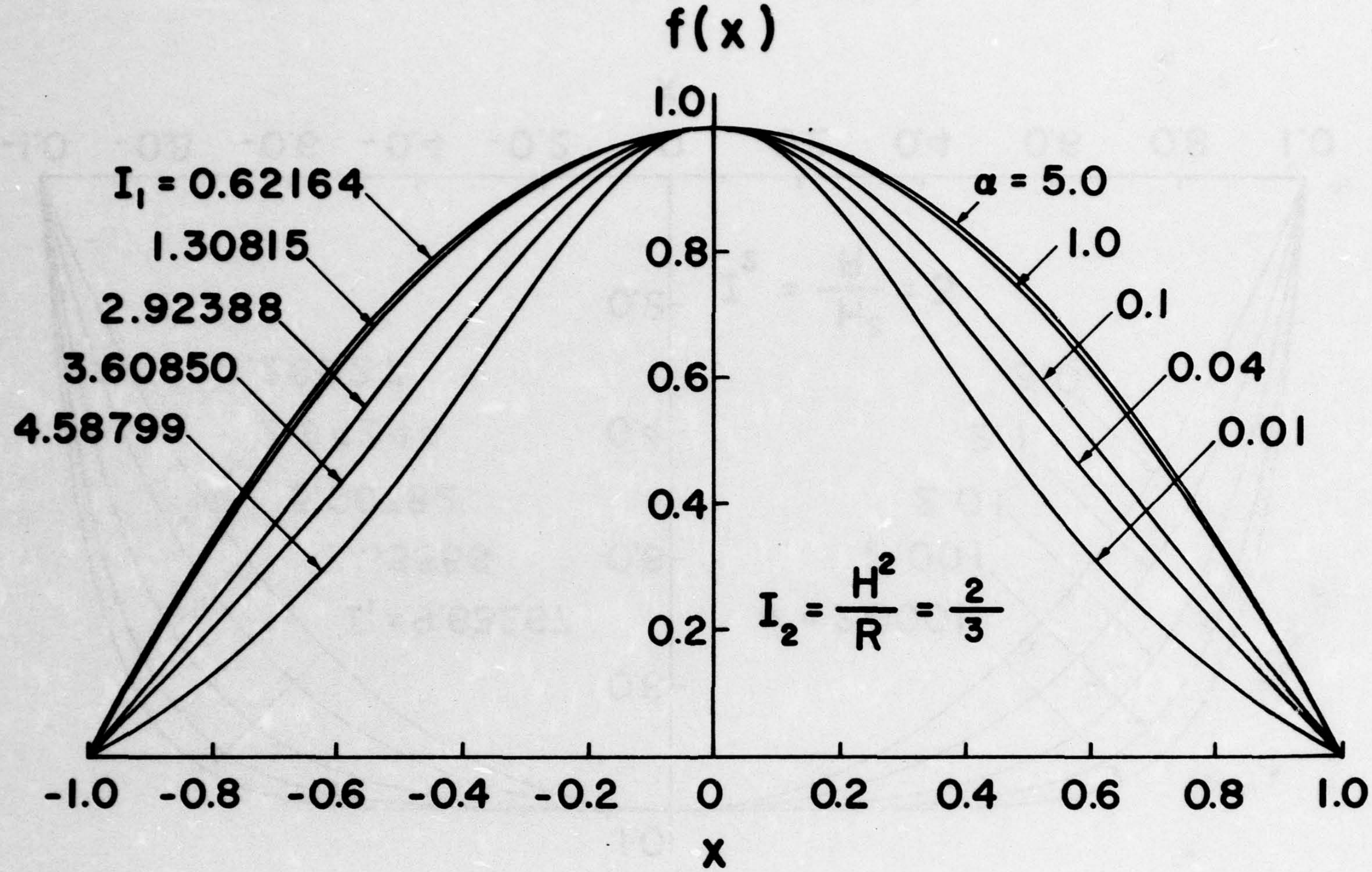

Fig. 5. Velocity amplitude $f(x)$ versus $x = \theta/\theta_o$ for $I_2 = \frac{2}{3}$ and various $I_1 = R^{1/2}\theta_o$.

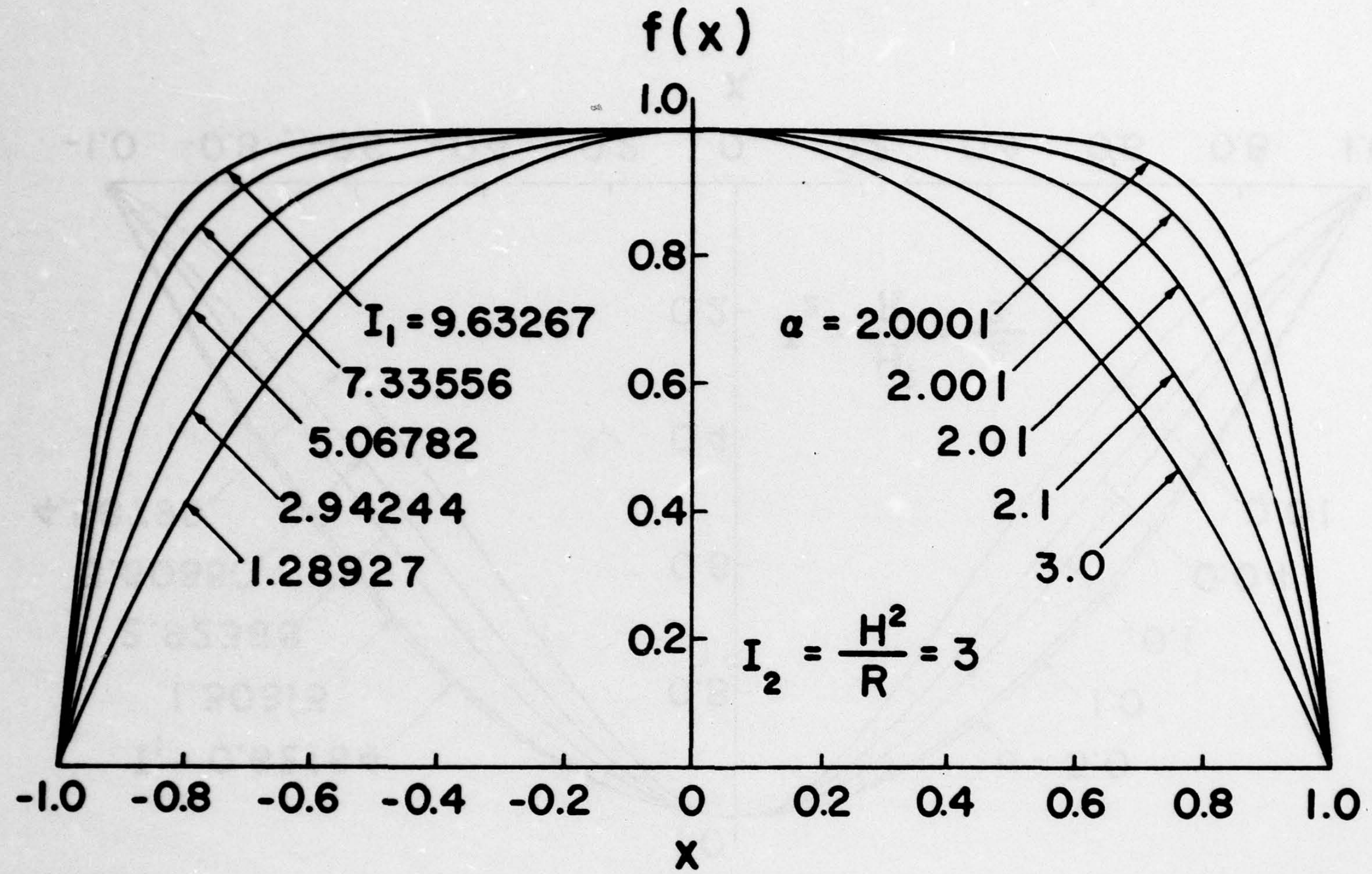

Fig. 6. Velocity amplitude $f(x)$ versus $x = \theta/\theta_o$ for $I_2 = 3$ and various $I_1 = R^{1/2}\theta_o$.

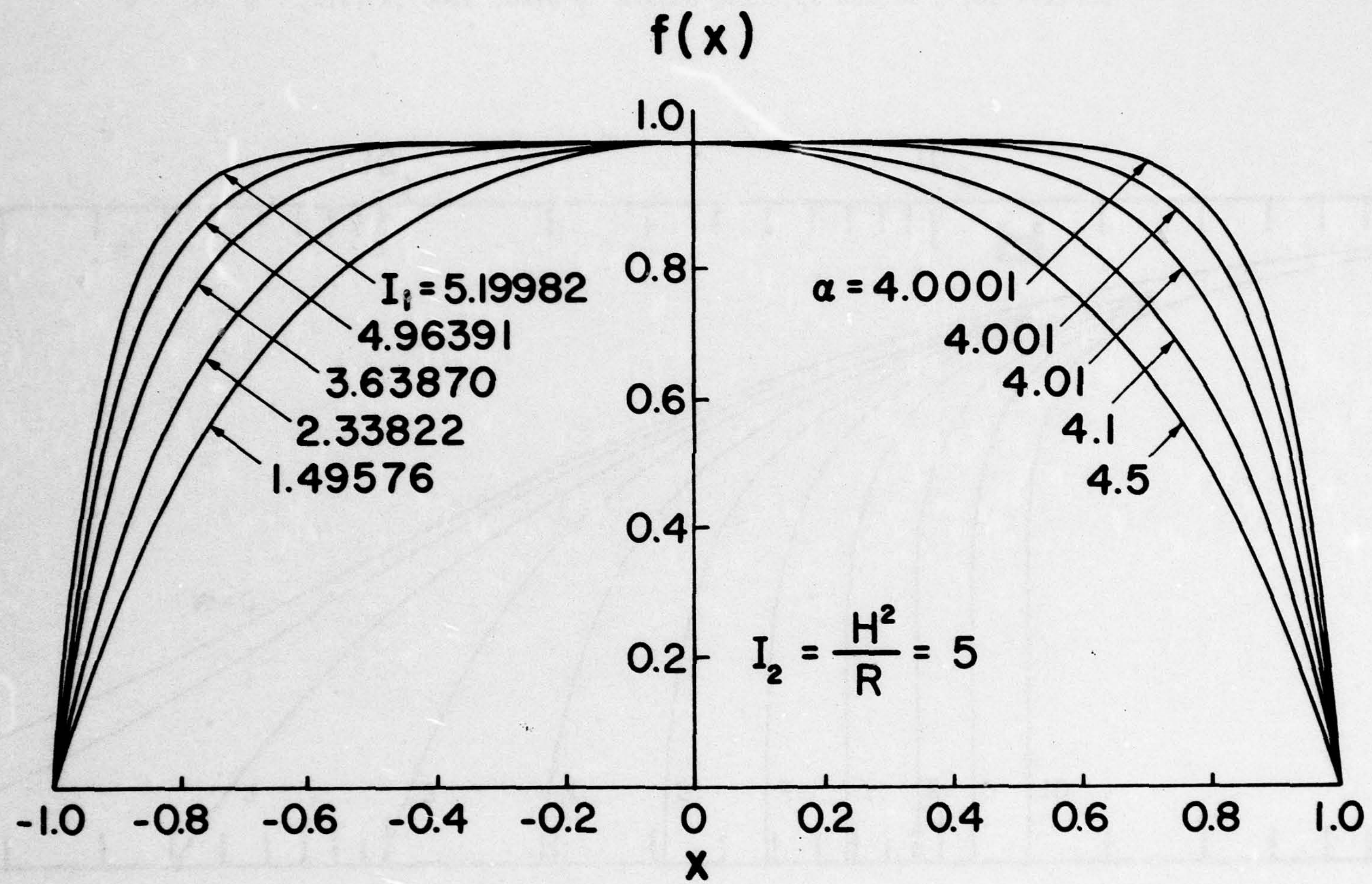

Fig. 7. Velocity amplitude $f(x)$ versus $x = \theta/\theta_o$ for $I_2 = 5$ and various $I_1 = R^{1/2}\theta_o$.

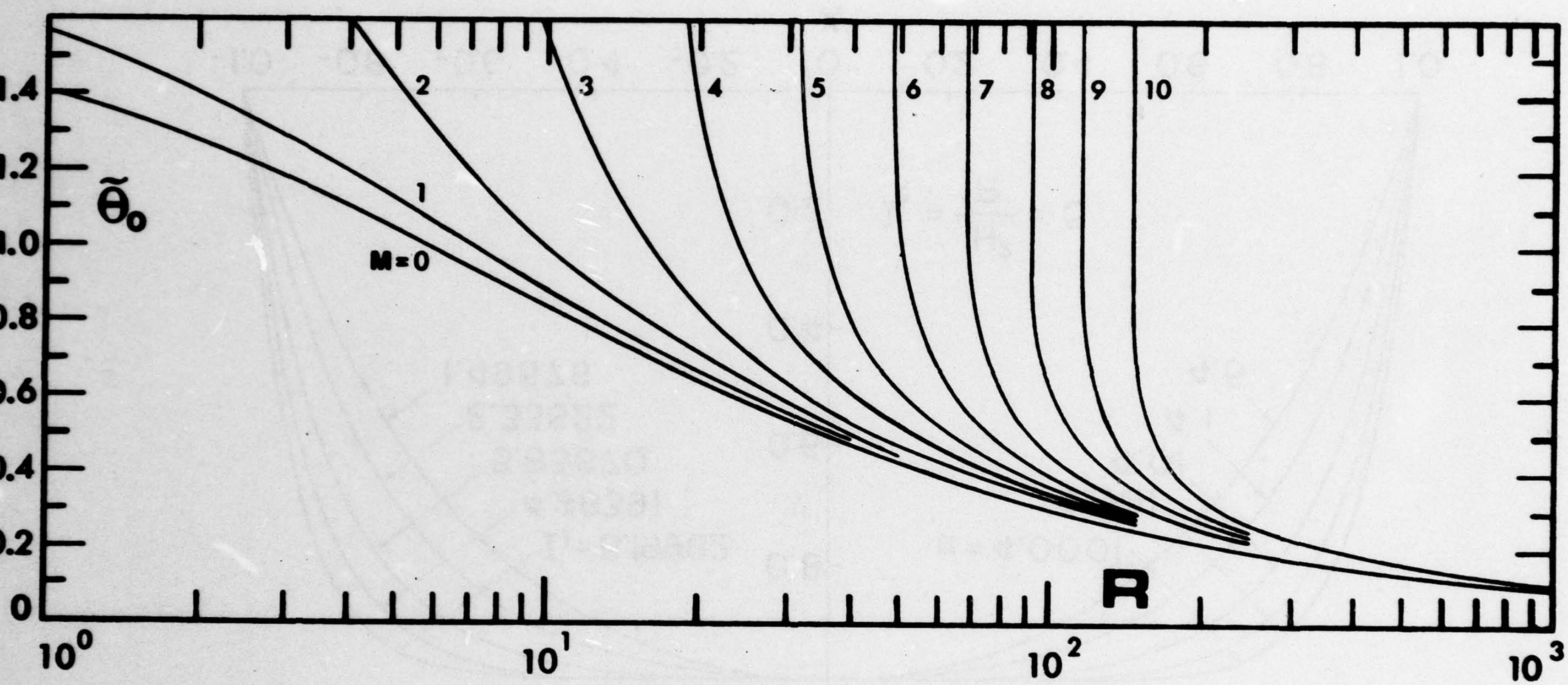

Fig. 8. Critical duct angle $\tilde{\theta}_o$ versus Reynolds number R for various Hartmann numbers H.

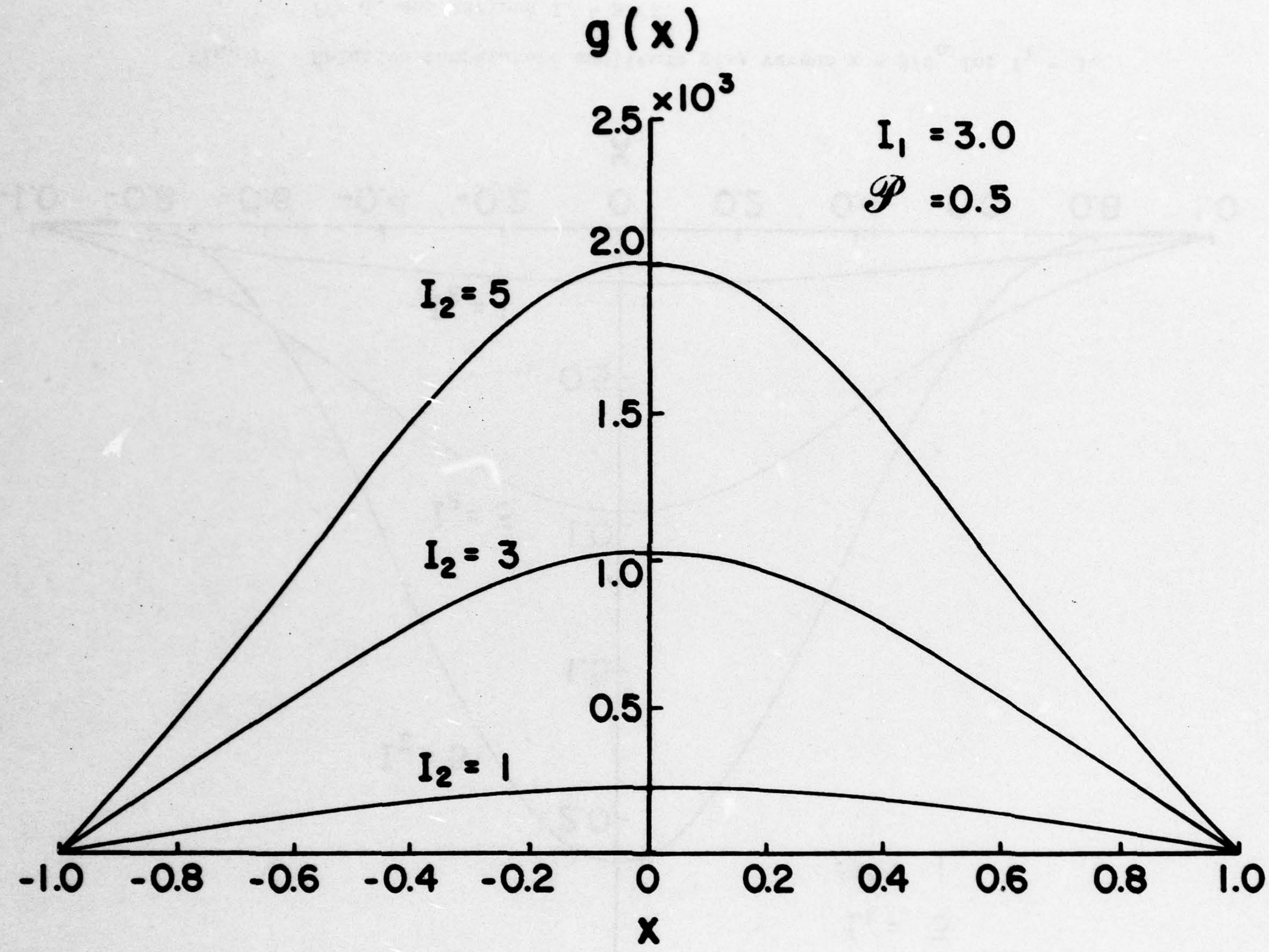

Fig. 9. Relative temperature amplitude $g(x)$ versus $x = \theta/\theta_o$ for $I_1 = 3$, $P = 0.5$, and various $I_2 = H^2/R$.

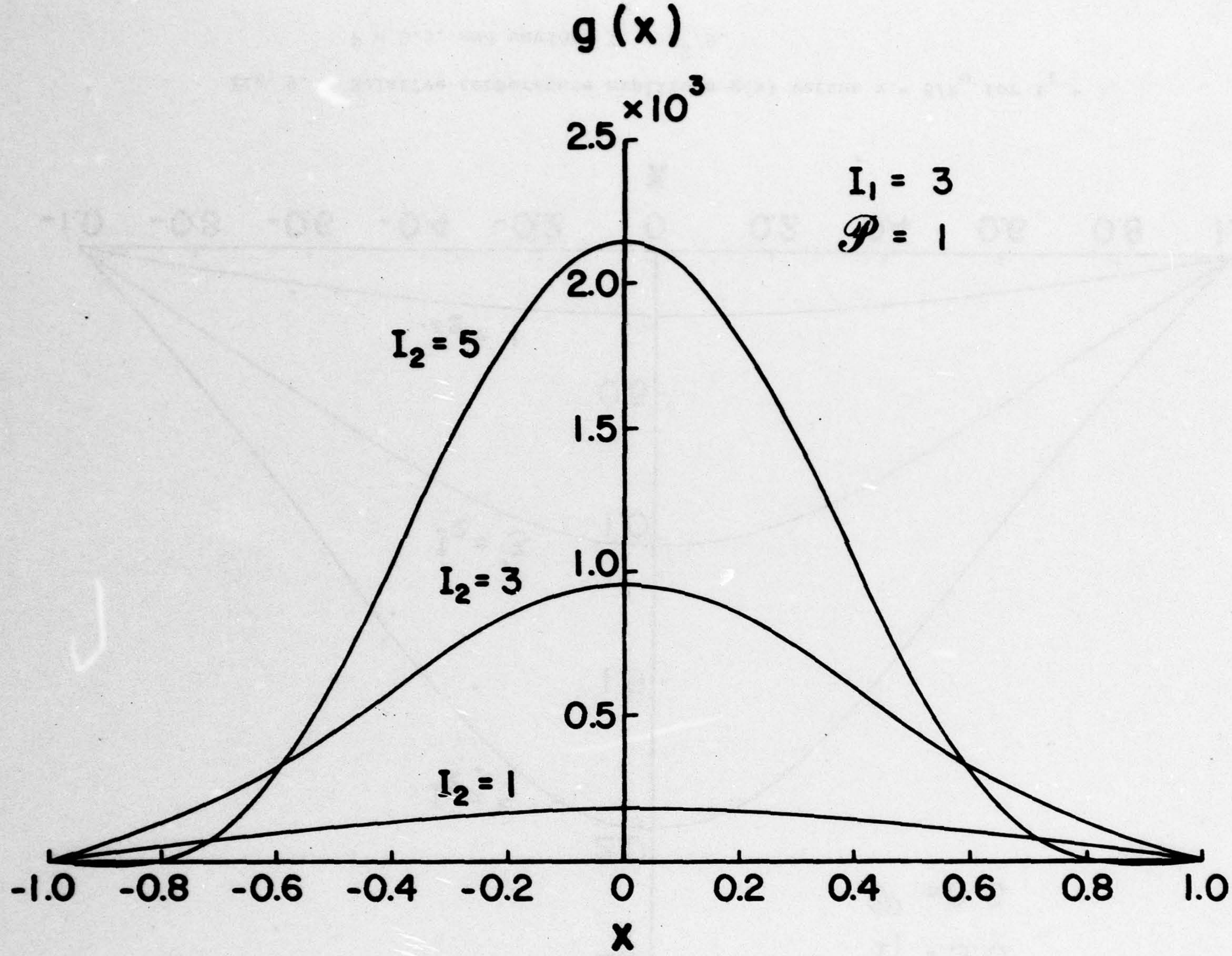

Fig. 10. Relative temperature amplitude $g(x)$ versus $x = \theta/\theta_o$ for $I_1 = 3$, $P = 1$, and various $I_2 = H^2/R$.

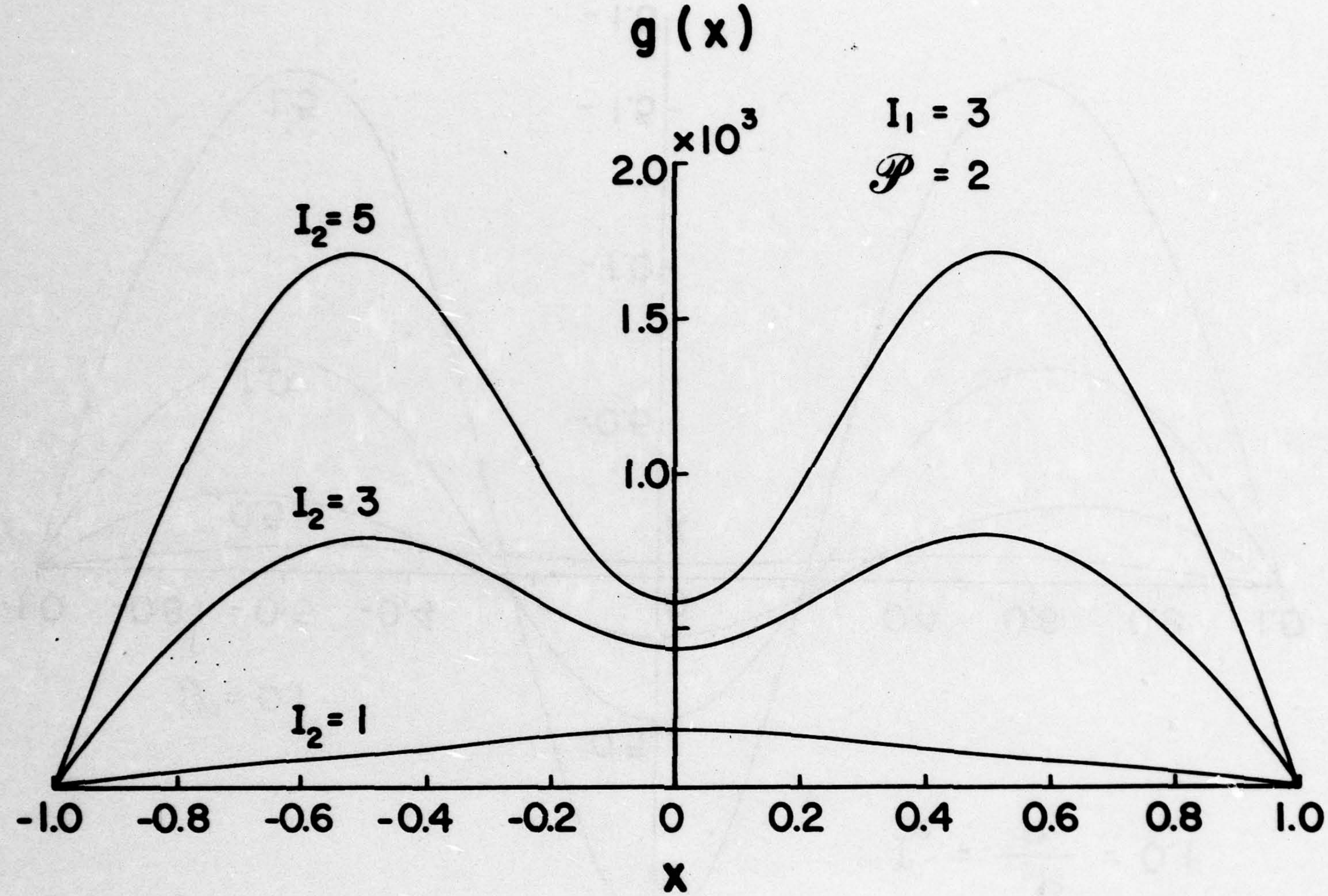

Fig. 11. Relative temperature amplitude g(x) versus $x = \theta/\theta_o$ for $I_1 = 3$, $\mathcal{P} = 2$, and various $I_2 = H^2/R$.

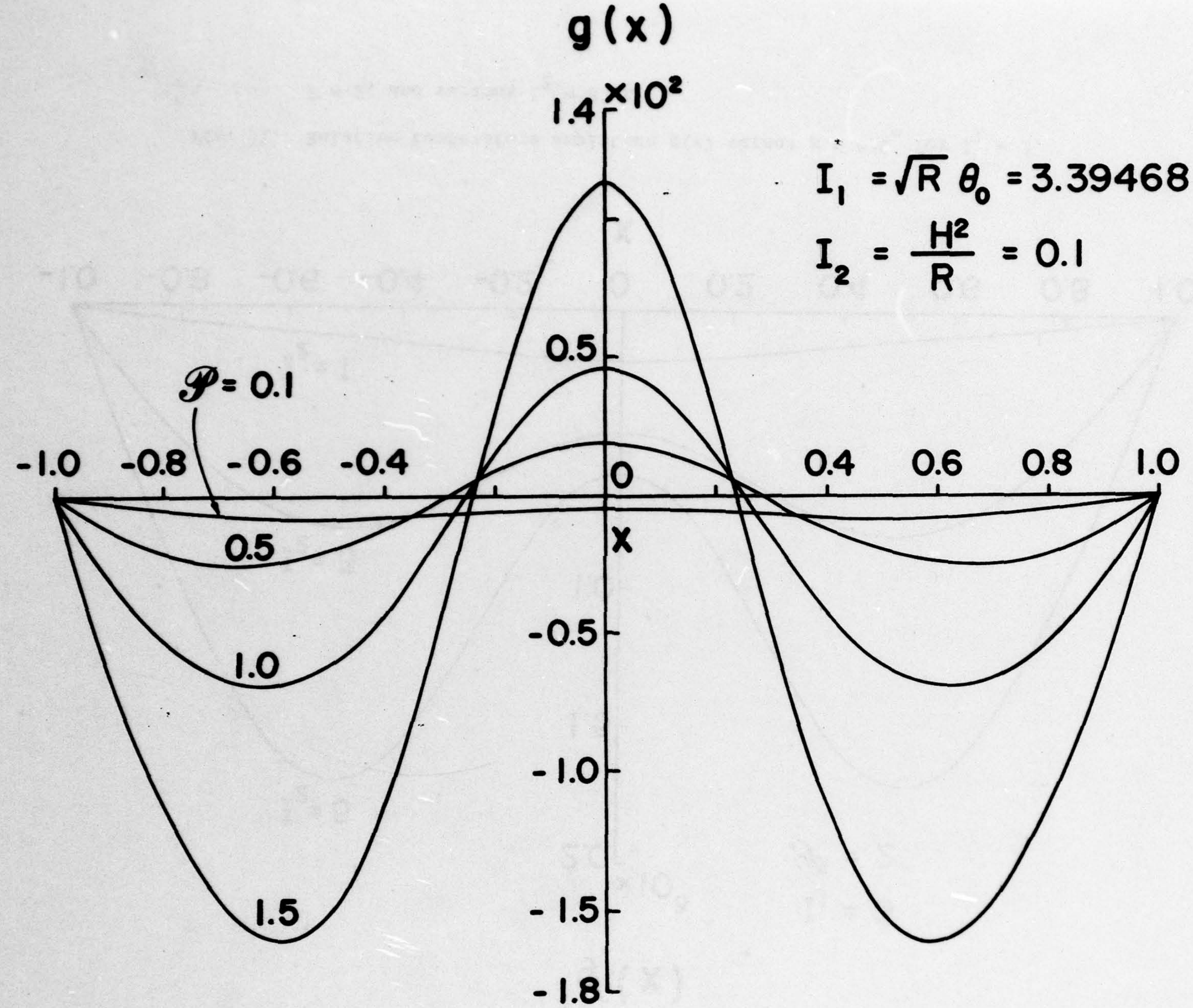

Fig. 12. Relative temperature amplitude g(x) versus $x = \theta/\theta_o$ for $I_1 = 3.39468$, $I_2 = 0.1$, and various Prandtl numbers P.

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11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY Office of Naval Research

13. ABSTRACT

The report contains theoretical work on ~~(1)~~ pressure ionization, ~~(2)~~ exchange energy, and ~~(3)~~ degeneracy effects in high pressure plasmas. In the numerical examples, alkali plasmas are discussed. In addition tables of the Sommerfeld inegrals of degenerate electron gases are presented. With regard to the acccleration of plasmas in high density plasmas, a critical analysis of the propagation of wave pockets is given within the frame of conventional (parabolic) quantum theory.

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